

## The Mathematics of the Pythagoreans

“Through intellectual and moral discipline, the human mind can arrive at the existence and properties of the mathematical forms, and then begin to unravel the mysteries of nature and the human soul.”

--R. Tarnas, The Passion of the Western Mind.<sup>1</sup>

Pythagoras, the Greek philosopher and mathematician best known for the famous Pythagorean Theorem, was interested in much more than just the proof of  $a^2+b^2 = c^2$ . Rather, he and his followers--known as the Pythagoreans--also explored the principles of mathematics, the concept of mathematical figures, and the idea of a proof (“Pythagoras of Samos” 3). Their method of exploration and the motivation behind it was to understand and come in “contact with a higher, a divine reality” (DeVogel 196). As a result, there was a mixture of mathematics and mysticism in the Pythagoreans’ geometry, arithmetic, astronomy, music, and theory of numbers (Hersh 92). Although the Pythagoreans were criticized by many who found fault with combining a higher truth with mathematics (Aristotle, for example, was critical of their confusion of the physical and metaphysical<sup>2</sup>), it cannot be denied that they were the first thinkers in the Western world to mathematicize the universe.<sup>3</sup> They interpreted the world “in terms of order and symmetry, based on fixed mathematical ratios” and “maintained that the cosmos was...expressible in terms of number” (Hersh 237; Philip 79). Due to this belief, and

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<sup>1</sup> Quote comes from: Hersh, Reuben. What is Mathematics, Really? New York: Oxford University Press, 1997

<sup>2</sup> Philip, J.A. Pythagoras and Early Pythagoreanism. Canada: University of Toronto Press, 1966. page 60.

<sup>3</sup> Weisstein, Erik W. Pythagoras of Samos. 1996-1999.  
<http://www.astro.virginia.edu/~eww6n/bios/Pythagoras.html>.

motivated by their mysticism, the Pythagoreans uncovered such ideas as the five regular solids, the Pythagorean Theorem, the existence of irrational numbers, and many other key ideas (Guthrie 329-330). Thus, by searching for a divine truth and universal harmony through mathematics, the Pythagoreans made many important discoveries in different areas of math and science that are both mathematically sound and continue to be applied today.

The Pythagoreans made numerous findings in the field of astronomy that are surprisingly accurate, considering the lack of observational instruments in their day. For example, they hypothesized that the earth is round by observing the shadow of the earth on the moon during a lunar eclipse. Also, they believed that the sun and other planets are spheres as well, and that they orbited each other and a “central fire” in a circular motion<sup>4</sup> (Weisstein 1). Furthermore, the Pythagoreans recognized that the “orbit of the moon was inclined to the equator of the Earth and that both the morning and evening star were the planet Venus ( “Pythaogras of Samos” 5; Weisstein 1). The metaphysical part of their astronomy relates to their belief in harmony. DeVogel writes: “[the] basic principle of a mathematically ordered universe [is that] all order is based on numerical proportion; that is why harmony is the universal, cosmic law” (163). An example of this belief is the Pythagoreans’ conjecture of the existence of ten planets. They could only observe nine, but hypothesized the existence of a tenth planet, a “counter-earth”--to them the number ten symbolized perfection and unity; due to their belief that the universe was harmonical, there had to be ten planets in order to fit their perception of the universe (DeVogel163; Philip 78). Also, the Pythagoreans believed that the planets generated sounds while

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<sup>4</sup> *Pythagoras of Samos*. <http://www.intelligentchild.com/astronomy/pythaogras.html>.

orbiting the universe, producing a “harmony of the spheres” (Weisstein 1). This leads into Pythagoras’ work on music and music theory.

Another great discovery attributed to Pythagoras is that “musical intervals may be reduced to numerical ratios” (DeVogel 162). He discovered three musical intervals, the fourth, the fifth, and the octave by experimenting with the monochord, a one-stringed instrument with a moveable bridge (Philip 126, Guthrie 24). Pythagoras observed that vibrating strings produce harmonious sounds when the “ratios of the lengths of the strings are whole numbers” ( “Pythaogras of Samos” 4). Specifically, the ratio of the octave is 1:2; the perfect fifth, 2:3; the perfect fourth, 3:4. (Guthrie 29). This mathematical and musical discovery is linked by the Pythagoreans to the universe itself; they considered the universe to be composed of numbers and thus concluded that the ratios that Pythagoras discovered in music were also present in the relations of celestial bodies to one another. As mentioned before, the Pythagoreans believed that the planets produced sounds while orbiting; furthermore, they applied the newfound ratios to the harmonious sounds that the planets supposedly produce (Philip 123). Again, this is another instance of applying mathematics to the universe, albeit metaphysically. More importantly, however, is that the discovery of musical intervals was the foundation of music theory and continues to be relevant.

The mathematical aspect of Pythagoreanism that is most metaphysical and subject to criticism is number theory—the belief that there is a deeper, divine significance to numbers. According to the Pythagoreans:

...number was the principle of a divine order in the Universe. The study of number and its laws therefore was the immediate contemplation of the divine Law by which everything is held together. (DeVogel 196)

The Pythagoreans attributed varying qualities to different numbers, each one being a “self-subsistent entity having a character and property of its own” (79). Also, the way in which numbers were combined was significant; for example, justice was considered to be a reciprocal relation between crime and punishment as well as a number squared ( Hersh 97).

A table of the integers from one to six and their meaning is seen below:

<u>Integer</u>	<u>Meaning</u> <sup>5</sup>
1	generator of numbers and number of reason
2	first even or female number
3	first true male number
4	justice or retribution
5	number of marriage (union of male and female)
6	number of creation

Most important is their belief that numbers were physical, existing entities that were the physical components of everything in existence. As Philip writes: “Everything is number” (61). They did not believe or reason that a number is just a number, for purposes of quantification. Nor did the Pythagoreans distinguish between the ideal and mathematical number (DeVogel 197; 204). Rather, their use of numbers was a search for truth and patterns in the universe; this search led them into true discoveries in mathematics and geometry.

“The Pythagoreans...were the first to use numerical and geometrical diagrams as models of cosmic wholeness and celestial order” (30). For example, they represented the number ten in the geometric form of the tetraktys (see figure below).

The Pythagoreans' method of arranging numbers geometrically may actually have been the way by which they discovered geometrical theorems (29). Another instance of arranging mathematical numbers into geometrical forms can be seen below:

The Pythagoreans are also credited with the construction of the five regular solids. Pythagoras probably constructed them out of right triangles (see below). Predictably, each solid had a symbolic meaning in relation to the Universe (314).

<u>Solid</u>	<u># of Sides</u>	<u>Symbolic Meaning</u>
Cube	6	Earth
Tetrahedron	4	Fire
Octahedron	8	Air
Dodecahedron	12	Aither
Icosahedron	20	Water

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<sup>5</sup> From Hersh page 93.

The Pythagoreans also tackled the idea of dimensions in terms of geometry. One represented the point . Two represented the line . Three represented the surface , and four was the tetrahedron , the first three-dimensional form (Guthrie 29). The single point was the generator of dimensions, the two points were a line of the first dimension, the three points not on a line were a triangle of the second dimension, and the four points not on a plane, the tetrahedron, had a volume of dimension three (Hersh 93). The Pythagoreans experimentation and formation of geometrical forms led them into significant mathematical discoveries because they recognized that numbers and geometry were linked.

As mentioned before, because the Pythagoreans tended to arrange numbers into geometrical shapes, they were inclined to make discoveries in the mathematical field of geometry. For example, they proved that the sum of the interior angles of a triangle is equal to 180 degrees by the following formula:

A polygon with  $n$  sides has sum of interior angles  $2n-4$  right angles.

Ex: a triangle has 3 sides

$$2(3)-4 = 2$$

$$2(90^\circ) = 180^\circ$$

This generalization is applicable to all polygons, in fact. It remains relevant, in our day, for purposes of construction, engineering, and applications of geometry. Also, they were aware of the properties of similar figures, and, as mentioned above, the five regular solids (Guthrie 329-330).

The Pythagoreans uncovered the existence of irrational numbers, as well. The concept of irrationality was anathema to a group of people who believed that numbers were commonsensical and upheld a harmonious universe, and yet they acknowledged and proved the irrationality of the  $\sqrt{2}$ . Again, they uncovered the irrationality through geometric means; they discovered that the diagonal of a square was incommensurable with its side<sup>6</sup>. Mathematically, this conjecture can be demonstrated by showing that there is “no pair of integers  $x$  and  $y$ , such that  $(x/y)^2 = 2$  (81)”. The Pythagoreans’ most famous discovery, however, was the Pythagorean Theorem. According to Euclid, the theorem states: “In any right triangle, the sum of the squares of the lengths of the two shorter sides equals the square of the length of the long side” (Hersh 272). Again, the Pythagoreans did not think of this in purely abstract means; they actually constructed squares on the sides of the triangles and calculated the areas of the squares, finding that the sum of the area of the two smaller squares was equal to the area of the larger square. This is in keeping with their geometrical way of viewing numbers and mathematics (apart from viewing the world metaphysically). See below for proof.

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<sup>6</sup> <http://www.scsd.k12.ny.us/levy/math/Pythagoras2.html>

Although the Pythagorean theorem was known to the Babylonians 1000 years earlier, Pythagoras was probably the first to prove it (“Pythagoras of Samos” 4). Due to this monumental proof, the idea can be applied to, among other things, construction (the pitch of a roof, the determination of whether or not a wall was perpendicular to the floor, et cetera), mathematics (such as the distance between two points on a graph), astronomy (measuring distances to and between stars), archaeology (measuring sites) and so on. The Pythagorean Theorem and other discoveries of the Pythagorean school are applied and relevant from the time the discoveries were made to the present day.

Thus, we see that, through Pythagoras’ and his followers’ search for truth in numbers and their formation of numbers into geometrical forms, many key mathematical ideas were uncovered that serve humans’ purposes to this day. Although the Pythagoreans were motivated less by a purely abstract quest for knowledge than motivated by a metaphysical desire to explain the universe, they nevertheless did uncover certain mathematical realities. As Aristotle wrote: “...the thinkers known as the Pythagoreans were the first to pursue mathematical studies and advance them” (Philip

78). As mathematical metaphysicists, they contributed many important ideas to mathematics.

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