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The Speed of a Ball on a Train

Suppose a stationary observer O is standing on the platform of a train station watching a train go by at a constant speed of v miles per hour. Furthermore, a passenger O' sees someone else on the train throw a ball to a friend, toward the front of the train, at u' miles per hour. What is the relativistic speed u of the ball according to the measurements of O?

To answer this question, first note that speed, being change in distance divided by change in time, requires consideration of the simultaneous relativity of time and space. Thus, we must use the Lorentz transformations to determine the speed of the ball according to O.

Let's suppose that O' determines that on the train the ball goes a distance x' miles in t' hours; so,

$$x' = u't'$$

Now, we use the Lorentz transformations to express x' and t' in terms of x and t, the corresponding distance and time measured on the platform by O. If we let

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

then substituting for x' and t' and solving for x in terms of t, we get:

$$\frac{x - vt}{\gamma} = u' \frac{t - \frac{vx}{c^2}}{\gamma}$$

$$x - vt = u'\left(t - \frac{vx}{c^2}\right)$$

$$x = vt + u't - u'\frac{vx}{c^2}$$

$$x\left(1 + \frac{vu'}{c^2}\right) = (v + u')t$$

$$x = \frac{v + u'}{1 + \frac{vu'}{c^2}}t$$

This last expression states that O on the platform computes the speed of the ball by first adding the speed of the train measured from the platform and the speed of the ball measured by O' on the train, and then dividing by $\left(1+\frac{vu'}{c^2}\right)$. That is, the speed of the ball from the platform is

$$u = \frac{v+u'}{1+\frac{vu'}{c^2}}$$

Note that if the product vu' of the speed of the train and the speed of the ball measured by O' is negligible when compared with the speed of light c, then the denominator is approximately equal to 1. In that case, we obtain the usual formula that O on the platform gets the speed of the ball by adding together the speed of the train and the speed of the ball measured by someone on the train.

For example, suppose a stationary observer O watches a train go by at a speed of 1/2 c, while O' on the train measures the speed of a ball thrown from one passenger to another toward the front of the train as 1/4 c. Then the speed of the ball according to O is:

$$\frac{\frac{c}{2}+\frac{c}{4}}{1+\frac{\left(\frac{c}{2}\right)\left(\frac{c}{4}\right)}{c^2}} = \frac{\frac{3c}{4}}{1+\frac{1}{8}} = \left(\frac{8}{9}\right)\left(\frac{3c}{4}\right) = \frac{2}{3}c$$

If we had made the calculation (incorrectly) without taking into account relativity, the answer would have been the sum of the two speeds, or $\frac{3}{4}c$.