Math 54 Final Exam

You may use and may only use Armstrong's text, your class notes, homework solutions which **you** wrote up, homework solutions placed on our web site, and your instructor for help. Prove all claims. You may use without proof any results proved in chapters 2-5 of the book and any results from assigned homework. Open ended questions should be answered with a proof or a counter example, where any counter example produced should be **proved** to be a counter example.

Part 1: Basic Concepts

Call a space locally compact if every point is contained in an open set whose closure is compact. Given a locally compact Hausdorff space X which is not compact we may form a new space $\check{X}$ which as a set is $X \cup \{\infty\}$ where $\{\infty\}$ is a set containing a single point denoted by ∞. Call a set open in $\check{X}$ if it is either an open set in X, all of $X \cup \{\infty\}$, or any set in the form $(X - K) \cup \{\infty\}$ with K a compact subset of X. $\check{X}$ is called the one point compactification of X.

For this problem let C be the infinite cylinder in $\mathbf{E^3}$ described by

$$\{(x, y, z) \in \mathbf{E}^3 \mid x^2 + y^2 = 1\},$$

and let O_4 be the disjoint union of four copies of the open interval $(0, 1)$.

1. (10 points) Prove the following.
   (a) The open sets described above indeed form a topology on $X \cup \{\infty\}$.
   (b) $\check{X}$ is a compact Hausdorff space.
   (c) X is dense in $\check{X}$.
   (d) The inclusion map of X into $\check{X}$ is an embedding.
   (e) Sketch what $\check{C}$ and $\check{O}_4$ will look like (by embedding them in $\mathbf{E}^3$).
2. (10 points) Let X and Y be locally compact Hausdorff space and prove the following.
   (a) A continuous map f from X to Y extends to a continuous map from $\check{X}$ to $\check{Y}$ if and only if $f^{-1}(K)$ is compact for every compact set K in Y.
   (b) If X and Y are homeomorphic then $\check{X}$ and $\check{Y}$ are homeomorphic.
   (c) Find two spaces which are not homeomorphic but have homeomorphic one point compactifications.

Part 2: Identification Spaces.

Let C_4 be the disjoint union of four copies of $[0, 1]$, and P_4 be the partition of C_4 into sets which includes the single points in the form $p \in (0, 1)$ for one of the copies of $[0, 1]$ and the set containing **all** the end points (i.e. "$\{0, 1, 0, 1, 0, 1, 0, 1\}$").

Let M be the"paper towel role" in $\mathbf{E}^3$ described by

$$\{(x, y, z) \in \mathbf{E}^3 \mid x^2 + y^2 = 1, -1 \le z \le 1\}.$$

Let P be the partition of this set which includes the singles points $(x, y, z) \in M$ when $|z| \neq 1$ and the set

$$\{(x, y, z) \in M \mid |z| = 1\}.$$

Let N be th regular octagon in $\mathbf{E}^2$ as pictured below, and let Q be the partition described by identifying points in the boundary as in the figure.

1. (10 points) Prove the following.
   (a) The identification space I_{P_4} is homeomorphic to $\check{O}_4$.
   (b) The identification space I_P is homeomorphic to $\check{C}$.
2. (5 points) Prove that I_Q is a compact surface.
3. (5 points) Prove that a retract is an identification map.

Part 3: The fundamental group and covering spaces.

1. (10 points) Find $\pi_1(S^2 - \{p, q\})$ where p and q are two distinct points in the sphere S^2.
2. (10 points) Prove a simply connected space can have only one path connected cover and describe this cover.
3. (10 points) Prove that $\pi_1(\check{O}_4)$ is not a commutative group.
4. (10 points) Using only what we know, prove I_Q is not homeomorphic to the sphere S^2.