Math 54 Exam

You may use and may only use Armstrong's text, your class notes, homework solutions which **you** wrote up, homework solutions placed on our web site, and your instructor for help. Prove all claims you make up, you may use any results proved in the book and any results from assigned homework problems. Open ended questions should be answered with a proof or a counter examples, where any counter example produced should be **proved** to be a counter example.

Part 1: Basic Concepts

In this section your ability to manipulate the basic concepts (topology, product spaces, connectedness, compactness, ...) will be tested by exploring these ideas in the presence of a metric.

A metric on a set M is a function

$$d : M \times M \to \mathbf{E}^1$$

such that for all $x, y, z \in M$ we have

(a) $d(x, y) \geq 0$ with equality iff $x = y$
(b) $d(x, y) = d(y, x)$
(c) $d(x, y) + d(y, z) \geq d(x, z)$

Let the metric topology on M be the smallest topology of M such that d is continuous (where $\mathbf{E}^1$ is the real line with its Euclidean topology and $M \times M$ is given the product topology). With this topology M is called a metric space. (Recall: To be the smallest topology on M such that d is continuos means that any topology where d is continuous will contain all of the metric topology's open sets.)

In the book metrics are discussed in section 2.4. I recommend reading over (or at least skimming over) section 2.4 and 3.3 before attempting these exercises. Notice, our definition of the metric topology here is slightly different from the book's definition and in problem 1 we will see why the definitions coincide.

1. (10 points) Let M be a metric space and let the open ball centered at x of radius ϵ be
$$\{y \in M \mid d(x, y) < \epsilon\}.$$
Let β be the collection of all the open balls (i.e. with every possible center and radius). Prove β is a basis for M's metric topology.

2. (5 points) Given a set X, is the function $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ defined on $X \times X$ a metric? If so describe the metric topology on X.

3. (5 points) A metric space M is called bounded if there is a real number C such that for any $x, y \in M$ we have that $d(x, y) < C$. Can a set M admit a pair of metrics which give rise to the **same** topology and have the property that one metric is bounded and other unbounded?

4. (5 points) If M and N are both metric spaces construct a metric on the product space $M \times N$ such that the metric topology agrees with the product topology on $M \times N$. Is this metric unique?

5. (5 points) Let X be a subset of $\{x_i\}_{i=1}^{\infty}$. Suppose X is a connected metric space and explicitly describe X.

6. (5 points) If A is a subset of a metric space M, let the diameter of A be defined by the supremum of $d(a,b)$ over all $a,b \in A$. If A is compact show that the diameter of A is finite and there are points $x,y \in M$ such that the diameter is equal to $d(x,y)$.

7. (5 points) Read lemma 3.11 and its proof carefully. Due problem 7 in section 3.3.

Part 2: Constructing examples.

Here your ability to construct topological spaces will be tested via exploring the solid torus and its relation to the three dimensional sphere.

Let CU be the cube in $\mathbf{E}^3$ with vertices in the set

$$\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$$

given the subspace topology as a topological space. We will be giving the $\mathbf{E}^3$ where CU lives the coordinates (θ,ψ,r) throughout this problem. Form the partition of CU whose elents are the intersection of CU with one of the following six types of subsets of $\mathbf{E}^3$:

$$\{(\theta,\psi,0) \mid 0 \le \psi \le 1, \theta \ne 0,1\}, \{(0,\psi,0),(1,\psi,0) \mid 0 \le \psi \le 1\},$$
$$\{(\theta,0,r),(\theta,1,r) \mid r \ne 0, \theta \ne 0,1\}, \{(0,\psi,r),(1,\psi,r) \mid r \ne 0; \psi \ne 0,1\},$$
$$\{(0,0,r),(0,1,r),(1,0,r),(1,1,r) \mid r \ne 0\}, or \{(\theta,\psi,r) \mid \theta,\psi \ne 0,1; r \ne 0\}.$$

Let ST be the identification space formed using this partition P, and call it the solid torus. Define its boundary to be the points where $r=1$.

View the three dimensional sphere, S^3, as

$$\{(z,w) \in \mathbf{C}\times\mathbf{C} \mid |z|^2+|w|^2=1\},$$

where $\mathbf{C}$ is the complex numbers.

1. (5 points) Here is a function from CU to $\mathbf{E}^3$.

   $$f(\theta,\psi,r) = (\cos(2\pi\theta)(2+r\cos(2\pi\psi)), \sin(2\pi\theta)(2+r\cos(2\pi\psi)), r\sin(2\pi\psi)).$$

   Prove that f induces an embedding of ST into $\mathbf{E}^3$ and describe its image.

2. (5 points) Here are a pair of functions from CU to S^3.

   $$g_1(\theta,\psi,r) = \left(\left(\sqrt{r/2}\right)e^{i2\pi\psi}, \left(\sqrt{1-r/2}\right)e^{i2\pi\theta}\right)$$

   and

   $$g_2(\theta,\psi,r) = \left(\left(\sqrt{1-r/2}\right)e^{i2\pi\theta}, \left(\sqrt{r/2}\right)e^{i2\pi\psi}\right).$$

   Prove that g_1 and g_2 both induce embeddings of ST into S^3.

3. (15 points) Show that you may take two copies of ST and identify the points on their boundaries in such a way as to form the three dimensional sphere.