Reference: Today we will apply some of the fundamental group ideas we have been developing. Here I collect the results we will be using and proving.

Some facts about the fundamental group $\pi_1(X, p)$:

1. **A Deck Like Action:** Given a topological space X and a group G given the discrete topology. A continuous action of G on X will be called deck like if for every $p \in X$ there is an open set $U \subset X$ such that $U \bigcap gU$ is empty for every $g \in G - id$.

   Example: The action of $\mathbf{Z}_2$ (which is isomorphic to $\{1, -1\}$ under multiplication) on $S^n = \{x \in \mathbf{E}^{n+1} \mid ||x|| = 1\}$ given by $-1 \cdot x = -x$ is a deck like action.

   Example: The action of the integers on $\mathbf{R}$ given by $m \cdot x = x + m$ is a deck like action.

2. **(The Deck Theorem)** If X is simply connected and G acts on X on a deck like way, then $\pi_1(X/G)$ is isomorphic to G.

   Example: We know that S^n is simply connected for $n > 1$, hence $\pi_1(P^n)$ is $\mathbf{Z}_2$.

   Example: We know that $\mathbf{R}$ is simply connected for $n > 1$, hence $\pi_1(S^1)$ is isomorphic to $\mathbf{Z}$.

3. **The Isomorphism:** Let the identification map be called $\pi : X \to X/G$. The isomorphism in the deck theorem is given by fixing $p \in X/G$ and $q \in \pi^{-1}(p)$ then defining $\Psi(g) =< \pi(\gamma_g) >$ with γ_g any path connecting q and $g \cdot q$.

4. **Mapping Properties:** A continuous map $f : X \to Y$ induces a homomorphism between fundamental groups $f_\star : \pi_1(X, p) \to \pi_1(Y, f(p))$ given by $f_\star(< \alpha >) =< f(\alpha) >$. Further this correspondence respects compositions, i.e. given a continuous map $g : Y \to Z$ we have that $g_\star \circ f_\star = (g \circ f)_\star$.

   Example: Let $f : P^n \to S^1$ be a continuous map. Then for any $< \alpha > \in \pi_1(P^n, p)$ we have that $< f(\alpha) >= id$ in $\pi_1(S^1, f(p))$.

Covering Spaces:

1. **Covering space definition and notation:** We say $\tilde{X}$ covers X if there is a continuous map $\pi : \tilde{X} \to X$ such that for every $p \in X$ there is an open set $V \subset X$ with $p \in V$ such that $\pi^{-1}(V) = \{U_\alpha\}$ with the U_α pairwise disjoint and satisfying the property that for each α we have that π restrict to U_α is a homeomorphism. In what follows, q will denote an element in $\pi^{-1}(p)$.

   Example: For any sphere S^n the mapping π sending x to the partition element $\{x, -x\}$ makes S^n into a cover of the projective plane P^n. Notice when $n = 1$ that we have a cover the circle with via another circle.

2. **(The path-lifting lemma)**: If γ is a path beginning at p, then there is a unique path $\tilde{\gamma}$ in $\tilde{X}$ which begins at q and satisfies $\pi \circ \tilde{\gamma} = \gamma$.

3. **(The homotopy-lifting lemma)**: If$F : I \times I \to X$ is a continuous map such that $F(0,t) = F(1,t) = p$ for all $0 \leq t \leq 1$, then there is a unique continuous map $\tilde{F} : I \times I \to \tilde{X}$ which satisfies $\pi \circ \tilde{F} = F$ and $\tilde{F}(0,t) = q$, for all $0 \leq t \leq 1$.

   Typical Use: Look at the previous example. Take any path $\tilde{\gamma}$ from q to $-q$ for for some $q \in S^n$ and define $\gamma = \pi \circ \tilde{\gamma}$. Notice that $\pi(q) = \pi(-q) = p \in P^n$ hence γ is a path and from the path lifting lemma $\tilde{\gamma}$ is indeed the unique lifting of the path γ to S^n starting at q. Use the homotopy lifting lemma to show that $< \gamma > \neq id$ in $\pi_1(P^n, p)$.

Nifty applications.

1. **(the Borsuk-Ulam Theorem)** Call a continuous mapping$f : S^n \to S^m$ antipode preserving if $\{f(x), f(-x)\} = \{f(x), -f(x)\}$ for every $x \in S^n$. Prove there is no antipode preserving mapping from S^2 to S^1.

2. **(The Meteorological Theorem)** Prove that for any continuous map $f : S^2 \to R^2$ that there is an $x \in S^2$ such that $f(x) = f(-x)$.

3. **(The Lusternik-Schirlmann Theorem)** Prove that if S^2 by $2 + 1$ closed set, then one of these sets contains an antipodal pair $\{x, -x\}$.

Retractions

1. **(Retractions)** If A is subspace of X then a continuous map $g : X \to A$ is called a retraction if $g(a) = a$ for all $a \in A$, and A is called a retract of X.

   Example: S^1 is a retract of $E^2 - \{(0,0)\}$.

2. **(Retraction Lemma)** Suppose A is retract of X, then the inclusion mapping of A into X induces a one to one homomorphism from $\pi_1(A, a)$ to $\pi_1(X, a)$.

   Example: S^1 is not a retract of the closed ball in $\mathbf{E}^2$, which we will denote B.

3. **(Brouwer Fixed Point Theorem)** Any continuous map of B to itself has a fixed point.

4. **(The "There's Your Scalp!" Theorem)** For every non-vanishing vector field V on B there is a point where V points directly inward and a point where V points directly outward.

A Rigorous Proof of a Great Theorem:

1. **(A Null Homotopy Lemma)** If a continuous map f of S^1 into X extends to a map of B, into X then $f_\star$ is trivial.

2. **(The Multiplication Mapping)** z^n restricted to S^1 is the multiplication by n map on $\pi_1(S^1)$.

3. **(An Extension Lemma)** A continuous map f of S^1 into X extends to a map of B into S^1 if and only if f is homotopic to a point.

4. **(The Fundamental Theorem of Algebra:)** Every polynomial equation $x^n + c_{n-1}x^{n-1} + \cdots + c_0$ of degree $n > 0$ has at least one root.

Separation Theorems

1. **(Tietze Extension Theorem)** Any real valued continuous function defined on a closed subset of a metric space can be extended to the whole space.

2. **(The path-lifting lemma)**: If γ is a path beginning at p, then there is a unique path $\tilde{\gamma}$ in $\tilde{X}$ which begins at q and satisfies $\pi \circ \tilde{\gamma} = \gamma$.

3. **(A non-separation theorem)** Suppose $f : I \to \mathbf{E}^2$ is an embedding. Then $\mathbf{E}^2 - f(I)$ has one component.

4. **(The Jordon Curve Theorem)** Suppose $f : S^1 \to \mathbf{E}^2$ is an embedding. Then $\mathbf{E}^2 - f(S^1)$ has exactly two components, the bounded and the unbounded.

5. **(Schoenflies Theorem)** Suppose $f : S^1 \subset \mathbf{E}^2 \to \mathbf{E}^2$ is an embedding, then f extends to a homeomorphism of $\mathbf{E}^2$ to $\mathbf{E}^2$. (In particular the bounded component is homeomorphic to a ball and the unbounded component homeomorphic to a ball minus a point).

6. **(A Fact)** The Jordon curve theorem is true in all dimensions but the Schoenflies theorem is not.