Solutions of homework problems
Day 9

Exercise 1 page 243

Prove the Leibnitz formula

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)\,dy = \int_{a(x)}^{b(x)} F_x(x,y)\,dy + F(x,b(x))\,b'(x) - F(x,a(x))\,a'(x)$$

Solution Put $I(x,a,b) = \int_a^b F(x,y)\,dy$

we have

$$\begin{tikzcd}
 & I \arrow[dl, "\frac{\partial I}{\partial x}"'] \arrow[d, "\frac{\partial I}{\partial a}"] \arrow[dr, "\frac{\partial I}{\partial b}"] & \\
x \arrow[d, "1=\frac{\partial x}{\partial x}"'] & a \arrow[d, "\frac{\partial a}{\partial x}"] & b \arrow[d, "\frac{\partial b}{\partial x}"] \\
x & x & x
\end{tikzcd}$$

$$\frac{dI}{dx} \underset{\text{chain rule}}{=} \underbrace{\frac{\partial I}{\partial x}}_{\int_a^b \frac{\partial F(x,y)}{\partial x}dy} \underbrace{\frac{dx}{dx}}_{=1} + (-1)\underbrace{F(x,a)}_{\text{FCT}}\frac{da}{dx} + (1)\,F(x,b)\frac{db}{dx} = \int_a^b F_x(x,y)\,dy + F(x,b(x))\,b'(x) - F(x,a(x))\,a'(x)$$

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Exercise 2.a page 243

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(a) find the eigenvalues and the orthonormal set of eigenvectors

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (5-\lambda)((1-\lambda)(4-\lambda)-4) =$$

expand with respect to last row

$$= (5-\lambda)(\cancel{4} - 5\lambda + \lambda^2 - \cancel{4}) = (5-\lambda)(\lambda-5)\lambda$$

Thus we have eigen values

$\lambda = 5$ of multiplicity two

$\lambda = 0$ of multiplicity 0

Let us find the eigenvector $\vec{e}_1$ corresponding to $\lambda_1 = 0$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) \qquad \vec{e}_1 = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \|\vec{e}_1\| = 1$$

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Let us find $\vec{e}_2, \vec{e}_3$ orthonormal corresponding to $\lambda_2 = 5$

$$\left(\begin{array}{ccc|c} -4 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{e}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) Solve in terms of eigenvectors

$$A\vec{x} - 2\vec{x} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

$$\vec{x} = c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$$

$$\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = f_1\vec{e}_1 + f_2\vec{e}_2 + f_3\vec{e}_3$$

$$1 = f_1\left(-\frac{2}{\sqrt{5}}\right) + f_2\left(\frac{1}{\sqrt{5}}\right) + f_3(0)$$

$$4 = f_1\left(\frac{1}{\sqrt{5}}\right) + f_2\left(\frac{2}{\sqrt{5}}\right) + f_3(0)$$

$$0 = f_1\left(\frac{1}{\sqrt{5}}\right) + f_2(0) + f_3(1) \Rightarrow f_3 = 0$$

$$\left(\begin{array}{cc|c} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & \frac{5}{\sqrt{5}} & 9 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 4 \end{array}\right)$$

$$f_3 = 0$$

$$f_2 = \frac{9}{\sqrt{5}} \qquad \frac{1}{\sqrt{5}} f_1 + \frac{2}{\sqrt{5}}\frac{9}{\sqrt{5}} = 4$$

$$f_1 = \frac{2}{\sqrt{5}} \qquad f_1 = \sqrt{5}\left(4 - \frac{18}{5}\right) = \sqrt{5}\,\frac{2}{5} = \frac{2}{\sqrt{5}}$$

$$(A-2I)(c_1\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3) = f_1\vec{e}_1+f_2\vec{e}_2+f_3\vec{e}_3$$

$$A(c_1\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3) - 2(c_1\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3) \; \text{"}$$

$$0c_1\vec{e}_1 + 5c_2\vec{e}_2 + 5c_3\vec{e}_3 - 2c_1\vec{e}_1 - 2c_2\vec{e}_2 - 2c_3\vec{e}_3$$

$$= f_1\vec{e}_1 + f_2\vec{e}_2 + f_3\vec{e}_3$$

$$\uparrow \quad \uparrow \quad \uparrow$$

linearly independent

$(0-2c_1) = f_1 \leftarrow$ coefficients in front of $\vec{e}_1$

$(5c_2 - 2c_2) = f_2 \leftarrow$ coefficient in front of $\vec{e}_2$

$(5c_3 - 2c_3) = f_3 \leftarrow$ coefficient in front of $\vec{e}_3$

$$c_1 = -\frac{1}{2} f_1 = -\frac{1}{2}\frac{2}{\sqrt{5}} = -\frac{1}{\sqrt{5}}$$

$$3c_2 = f_2 \Rightarrow c_2 = \frac{1}{3} f_2 = \frac{1}{3}\frac{2}{\sqrt{5}} = \frac{2}{3\sqrt{5}}$$

$$3c_3 = f_3 = 0 \Rightarrow c_3 = 0$$

Thus $\vec{x} = \left(-\frac{1}{\sqrt{5}}\right)\begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} + \frac{2}{3\sqrt{5}}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Problem 3 page 244

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

Find conditions on $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ s.t. the system $(A - 4I)\vec{u} = \vec{b}$ has a solution.

Solution exists when $\vec{b}$ is in the image of the linear operator $(A - 4I)$

$$(A - 4I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= \begin{pmatrix} -2v_1 + 2v_2 \\ 1v_1 + (-1)v_2 \end{pmatrix} = \begin{pmatrix} 2(v_2 - v_1) \\ -1(v_2 - v_1) \end{pmatrix} = (v_2 - v_1)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Thus the solution exists exactly when $\vec{b} = k\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. In this case for every chosen $\vec{b}$ you will have infinitely many different solutions