Math 46

Sketches of solutions of homework problems Day 8

Problem 3 page 225

Find the eigenvalues and eigenfunctions for the problem with periodic boundary conditions

$$-y''(x) = \lambda y(x) \qquad 0 \le x \le \ell$$

$y(0) = y(\ell)$

$y'(0) = y'(\ell)$ ← note that these are not the standard conditions in SLP

but they also are homogenous so we should have vector spaces of solutions

Case A $\lambda = \mu^2 > 0$ Case B $\lambda = 0$

Case C $\lambda = -\mu^2 < 0$

Case A $-y'' = \mu^2 y \Rightarrow y'' + \mu^2 y = 0$

characteristic equation $r^2 + \mu^2 = 0$

$r_{1,2} = \pm i\mu$. Thus the fundamental solution set is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

$$y(0) = c_1 \overset{?}{=} c_1 \cos(\mu \ell) + c_2 \sin(\mu \ell) = y(\ell)$$

$$y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$$

$$y'(0) = c_2\mu \overset{?}{=} \underbrace{-c_1\mu\sin(\mu\ell) + c_2\mu\cos(\mu\ell)}_{= y'(\ell)}$$

Thus

$$\left.\begin{aligned} c_1 &= c_1\cos(\mu\ell) + c_2\sin(\mu\ell) \\ c_2\mu &= -c_1\mu\sin(\mu\ell) + c_2\mu\cos(\mu\ell) \end{aligned}\right\} \begin{matrix} (a) \\ (b) \end{matrix}$$

multiply (a) by $c_2\mu$
(b) by c_1 and subtract

$$\underbrace{c_1c_2\mu - c_2c_1\mu}_{=0} = c_1c_2\mu\cancel{\cos(\mu\ell)} + c_2^2\mu\sin(\mu\ell) + c_1^2\mu\sin(\mu\ell) - c_1c_2\mu\cancel{\cos(\mu\ell)}$$

$$\Rightarrow (c_1^2 + c_2^2)\mu\sin(\mu\ell) = 0$$

$c_1^2 + c_2^2 \neq 0 \quad \mu > 0 \Rightarrow \mu = \frac{\pi n}{\ell} \quad n \in \mathbb{Z}$

$\lambda = \mu^2$ so WLOG $n \in \mathbb{N}$

is (a) true

$$c_1 \overset{?}{=} c_1 \underbrace{\cos\left(\frac{\pi n}{\ell}\ell\right)}_{=(-1)^n} + c_2 \overset{=0}{\sin\left(\frac{\pi n}{\ell}\ell\right)}$$

so n has to be <u>even</u>

(b)

$$c_2\frac{\pi n}{\ell} = -c_1\frac{\pi n}{\ell}\overset{=0}{\sin\left(\frac{\pi n}{\ell}\ell\right)} + c_2\frac{\pi n}{\ell}\overset{=(-1)^n}{\cos\left(\frac{\pi n}{\ell}\ell\right)}$$

is true if n is even

Thus we found eigen values

$2^2, 4^2, \ldots (2n)^2 \longleftrightarrow \cos\left(\frac{2n\pi}{\ell}x\right)$

$\cos\left(\frac{2\pi}{\ell}x\right)$ $\cos\left(\frac{4\pi}{\ell}x\right)$

$\lambda_n = (2n)^2$ $y_n = \cos\left(\frac{2n\pi}{\ell}x\right)$

Case B $\lambda = 0$

$y(x) = Ax + B$

$y(0) = y(\ell) \Rightarrow A = 0$

$y'(0) = y'(0) \Rightarrow B$ is anything

Thus $\lambda = 0$ is an eigenvalue corresponding to the eigenfunction $y = 1$ note that this can be written as $\cos\left(\frac{2 \cdot 0 \cdot \pi}{\ell}x\right)$

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case C $\lambda = -\mu^2 < 0$

$-y'' = -\mu^2 y$

$y'' - \mu^2 y = 0$ $r^2 - \mu^2 = 0$ is the characteristic equation

$y(t) = c_1 e^{\mu x} + c_2 e^{-\mu x}$ is the fundamental solution set

$y(0) = c_1 \cdot 1 + c_2 \cdot 1 \stackrel{?}{=} y(\ell) = c_1 e^{\mu \ell} + c_2 e^{-\mu \ell}$

$y'(x) = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}$

$y'(0) = c_1 \mu - c_2 \mu \stackrel{?}{=} c_1 \mu e^{\mu \ell} - c_2 \mu e^{-\mu \ell}$ " $y'(\ell)$

Thus we have

$c_1 + c_2 = c_1 e^{\mu \ell} + c_2 e^{-\mu \ell}$ (a)

$c_1 \mu - c_2 \mu = c_1 \mu e^{\mu \ell} - c_2 \mu e^{-\mu \ell}$ (b)

multiply (a) by (μ) and add to (b)

$2 c_1 \mu + \cancel{c_2 \mu} - \cancel{c_2 \mu} = c_1 \mu e^{\mu \ell} + c_1 \mu e^{\mu \ell} + \cancel{c_2 \mu e^{-\mu \ell}} - \cancel{c_2 \mu e^{-\mu \ell}}$

$2 c_1 \mu = 2 c_1 \mu e^{\mu \ell} \Rightarrow \mu \ell = 0 \Rightarrow \mu = 0$

This is impossible since $\mu \neq 0$

$\lambda = -\mu^2$

The answer is

The eigenvalues are

$\lambda_n = (2n)^2$

$n = 0, 1, 2, 3, \ldots$ ← note how the zero eigenvalue was included into this formula ☺

The eigenfunctions are

$y_n(x) = \cos\left(\frac{2\pi n x}{\ell}\right)$

Problem 4 page 225

$$-y'' = \lambda y \qquad 0<x<1$$

$$y(0)+y'(0)=0 \qquad y(1)=0$$

Solution

Is $\lambda=0$ an eigen value?

When $\lambda=0$ we get $-y''=0$

So the solution is $y(x)=Ax+B$

$$\left.\begin{aligned} y(0)+y'(0)=0 &\Rightarrow B+A=0 \\ y(1)=0 &\Rightarrow A+B=0 \end{aligned}\right\} A=-B$$

Thus $y(x) = x-1$ is an eigen function corresponding to $\underset{\substack{\uparrow \\ \text{eigenvalue}}}{\lambda}=0$

Are there any negative eigenvalues

$\lambda=-\mu^2 \quad \mu>0 \qquad -y''=-\mu^2 y \Rightarrow y''-\mu^2 y=0$

The characteristic equation

$r^2-\mu^2=0 \qquad r_{1,2}=\pm\mu$

$y(x)=c_1e^{\mu x}+c_2e^{-\mu x}$ is the fundamental solution

$$\left.\begin{aligned} y(0)+y'(0)=0 &\Rightarrow c_1+c_2+c_1\mu-c_2\mu=0 \quad (a) \\ y(1)=0 &\Rightarrow c_1e^{\mu}+c_2e^{-\mu}=0 \quad (b) \end{aligned}\right\}$$

(a) $c_1(1+\mu) + c_2(1-\mu) = 0 \Rightarrow$

From (b) $c_1, c_2 \neq 0$

$$c_1 = -c_2 \frac{(1-\mu)}{1+\mu}$$

(b) $$-c_2 \frac{(1-\mu)}{1+\mu} e^{\mu} + c_2 e^{-\mu} = 0$$

$$c_2 e^{\mu}\left(\underbrace{-\frac{1-\mu}{1+\mu}}_{>-1} + \underbrace{e^{-2\mu}}_{>1}\right)$$

since $\mu > 0$

so this term is never zero and there are <u>no</u> negative eigen values.

positive eigenvalues

$-y'' = \mu^2 y \qquad \mu > 0$

$y(0) + y'(0) = 0$

$y(1) = 0$

The characteristic equation is

$-r^2 = \mu^2 \qquad r_{1,2} = \pm i\mu$

$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$ is the fundamental solution.

$y(0) + y'(0) = 0 \Rightarrow$

$c_1 + \mu c_2 = 0$

$c_1 \cos(0) + c_2 \sin(0) - c_1 \mu \sin(0) + c_2 \mu \cos(0) = 0$

$y(1)=0 \Rightarrow c_1\cos\mu + c_2\sin\mu = 0$

Thus
$$\left.\begin{aligned} c_1+\mu c_2 &= 0 \\ c_1\cos\mu + c_2\sin\mu &= 0 \end{aligned}\right\} \begin{matrix} \text{(a)} \\ \text{(b)} \end{matrix}$$

$$-\mu c_2\cos\mu + c_2\sin\mu = 0$$

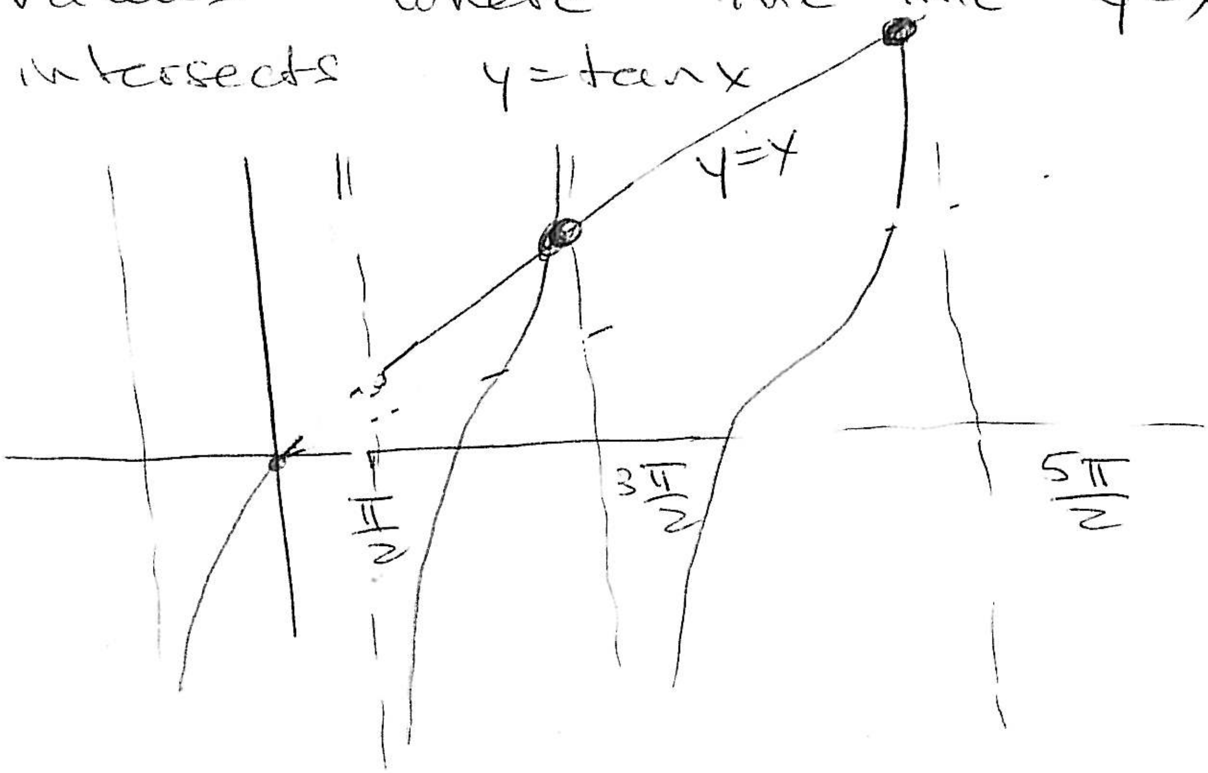
If $c_2=0 \Rightarrow c_1=0$ by (a) and we do not have eigen functions

Thus wlog $c_2\neq 0$

$$c_2(-\mu\cos\mu+\sin\mu)=0$$

$$\mu=\tan\mu$$

And for μ we can take any of the infinitely many positive values where the line $y=x$ intersects $y=\tan x$



Problem 7 page 225

$$-(x^2y')' = \lambda y \qquad 1 < x < e$$
$$y(1) = y(e) = 0$$

Use energy argument to show that eigenvalues must be nonnegative. Multiply by $y(x)$

$$-(x^2y')'y = \lambda y^2$$ and integrate on $[1, e]$

$$\int_1^e -(x^2y')'y\,dx = \int_1^e \lambda y^2$$

$\parallel \leftarrow$ by parts

$$\underbrace{-x^2y'y\Big]_{x=1}^{x=e}}_{\substack{\parallel \\ 0}} + \int_1^e x^2y'y'\,dx$$

0 since $y(1)=y(e)=0$

Thus $$\underbrace{\int_1^e x^2(y')^2\,dx}_{\geq 0} = \lambda \underbrace{\int_1^e y^2\,dx}_{> 0}$$ $\leftarrow$ since $y \equiv 0$ is not an eigen function

So λ has to be ≥ 0 to make this work

Note that $\lambda = 0$ actually works for $y(x) = C$