Solutions to Math 46 homework exercises Day 6

<u>Exercise</u> Use the Cauchy Schwarz inequality to show that $L^2[a,b]$ is a vector space.

<u>Solution</u> We need to show that

(a) $0 \in L^2$ (b) if $f \in L^2$, $\alpha \in \mathbb{R} \Rightarrow \alpha f \in L^2$

(c) if $f, g \in L^2 \Rightarrow f+g \in L^2$

(a) $\int_a^b |0|^2 dx = 0 < \infty$

(b) $\int_a^b |f|^2 dx < \infty$, $\int_a^b |(\alpha f)|^2 dx = \alpha^2 \int_a^b |f|^2 dx < \infty$

(c) $\int_a^b |f|^2 dx < \infty$ $\int_a^b |g|^2 dx < \infty$ we need to show that $\int_a^b |(f+g)|^2 dx < \infty$

$$\int_a^b (f+g)^2 dx = \underbrace{\int_a^b f^2 dx}_{<\infty} + \underbrace{\int_a^b g^2 dx}_{<\infty} +$$

$+ \; 2\int_a^b fg\,dx$. Thus it suffices to show that $\left|\int_a^b fg\,dx\right| < \infty$

$$\left|\int_a^b fg\,dx\right| = |(f,g)| \underset{\text{Cauchy Schwarz}}{\leq} \underbrace{\|f\|}_{<\infty}\;\underbrace{\|g\|}_{<\infty} < \infty$$

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By Exercise 4

$P_0(x) = 1$

$$P_1(x) = x - \frac{(x, P_0)}{(P_0, P_0)} P_0 = x - \frac{\int_{-1}^{1} x \cdot 1\,dx}{\int_{-1}^{1} 1 \cdot 1\,dx} \cdot 1 =$$

$$= x - 0 = x$$

$$P_2(x) = x^2 - \frac{(x^2, x)}{(x, x)} x - \frac{(x^2, 1)}{(1, 1)} \cdot 1 =$$

$$= x^2 - \left(\frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} \right) dx - \frac{\int_{-1}^{1} x^2 \cdot 1\,dx}{\int_{-1}^{1} 1 \cdot 1\,dx} \cdot 1 =$$

$$= x^2 - \frac{\frac{2}{3}}{2} \cdot 1 = x^2 - \frac{1}{3}$$

$$P_3(x) = x^3 - \frac{(x^3, x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} (x^2 - \tfrac{1}{3}) -$$

$$- \frac{(x^3, x)}{(x, x)} x - \frac{(x^3, 1)}{(1, 1)} \cdot 1 =$$

$$= \frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{3}\right)} x$$

$$= x^3 - \frac{\int_{-1}^{1} x^3 (x^2 - \frac{1}{3})\,dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2\,dx} - \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx} x - \,=$$

$$= x^3 - \frac{3}{5} x \quad = 0 \quad - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} 1^2\,dx} \cdot 1 \quad = 0$$

$P_0(x) = 1 \quad P_1(x) = x \,,\quad P_2(x) = x^2 - \frac{1}{3}$

$P_3(x) = x^3 - \frac{3}{5}x$

$e^x \approx c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \cdots$

where $c_0 = \dfrac{(e^x, P_0)}{(P_0, P_0)}$

$c_1 = \dfrac{(e^x, P_1)}{(P_1, P_1)}$

$c_2 = \dfrac{(e^x, P_2)}{(P_2, P_2)}$

$$c_0 = \frac{\int_{-1}^{1} e^x \cdot 1\,dx}{\int_{-1}^{1} 1 \cdot 1\,dx} = \frac{e - e^{-1}}{2}$$

$$c_1 = \frac{\int_{-1}^{1} e^x \cdot x\,dx}{\int_{-1}^{1} x \cdot x\,dx} = \frac{e^x \cdot x\Big]_{x=-1}^{x=1} - \int_{-1}^{1} e^x\,dx}{\frac{2}{3}} =$$

$$= \frac{(e + e^{-1}) - (e - e^{-1})}{\frac{2}{3}} = \frac{2e^{-1}}{\frac{2}{3}} = 3e^{-1}$$

$$c_2 = \frac{\int_{-1}^{1} e^x\left(x^2-\frac{1}{3}\right)dx}{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2dx}$$

$$\int_{-1}^{1}\underbrace{\left(x^2-\frac{1}{3}\right)^2}_{\uparrow\ \text{even}}dx = 2\int_{0}^{1} x^4-\frac{2}{3}x^2+\frac{1}{9}\,dx =$$

$$= 2\frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x\Big]_{x=0}^{x=1} =$$

$$= \frac{2}{5} - \frac{2}{9} + \frac{1}{9} = \frac{2^{(9}}{5} - \frac{1^{(2}}{9} = \frac{16}{45}$$

$$\int_{-1}^{1} e^x\left(x^2-\frac{1}{3}\right)dx = e^x\left(x^2-\frac{1}{3}\right)\Big]_{x=-1}^{x=1} -$$

$$-\int_{-1}^{1} e^x\cdot 2x\,dx = e\left(\frac{2}{3}\right) - e^{-1}\left(\frac{2}{3}\right) -$$

$$-2\underbrace{\int_{-1}^{1} e^x\cdot x\,dx}_{2e^{-1}} = \frac{2}{3}\left(e-e^{-1}\right) - 4e^{-1} =$$

$$= \frac{2}{3}e - \frac{14}{3}e^{-1}$$

$$e^x \approx \left(\frac{e-e^{-1}}{2}\right)\cdot 1 + 3e^{-1}x + \left(\frac{2}{3}e - \frac{14}{3}e^{-1}\right)\left(x^2-\frac{1}{3}\right)$$

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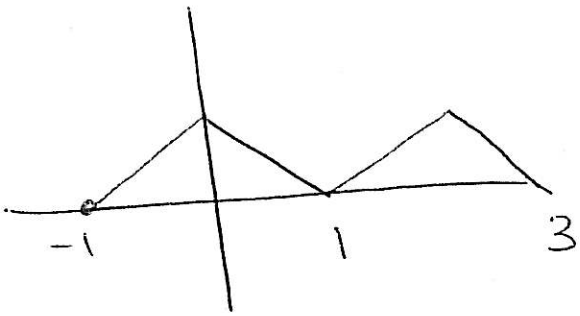
$$f(x) = \begin{cases} x+1 & \text{if } -1<x\leq 0 \\ 1-x & \text{if } 0<x\leq 1 \end{cases}$$



The function is even so it will have only the cosine terms

$$a_0 = \frac{1}{1}\int_{-1}^{1} \underbrace{f(x)\cdot 1}_{\text{even}}\, dx = 2\int_0^1 f(x)\,dx = 2\cdot\frac{1}{2} = 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} \underbrace{f(x)\cos\left(\frac{n\pi x}{1}\right)}_{\text{even}} dx =$$

$$= 2\int_0^1 f(x)\cos\left(\frac{n\pi x}{1}\right) dx =$$

$$= 2\int_0^1 (1-x)\underbrace{\cos(n\pi x)}_{\left(\frac{1}{n\pi}\sin(n\pi x)\right)'} dx =$$

$$= \underbrace{2(1-x)\frac{1}{n\pi}\sin(n\pi x)\Big]_{x=0}^{x=1}}_{=0} -$$

$$-2\int_0^1 \frac{1}{n\pi}\sin\left(\frac{n\pi x}{1}\right)\underbrace{(-1)}_{\substack{\| \\ (1-x)'}}\,dx =$$

$$= 2\int_0^1 \frac{1}{n\pi}\sin(n\pi x)\,dx =$$

$$= -\frac{2}{(n\pi)^2}\cos(n\pi x)\Big]_{x=0}^{x=1} =$$

$$= \frac{-2}{(n\pi)^2}\left((-1)^n-1\right)$$

Thus the Fourier series is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{1}\right) =$$

$$= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{-2}{(n\pi)^2}\left((-1)^n-1\right)\cos(n\pi x)$$

$\uparrow$ zero if n is even

$\uparrow$ you can leave the answer as is or you can simplify further to get

$(-2)(-2)$ if n is odd

$$\frac{1}{2}+\sum_{k=0}^{\infty}\frac{4}{\left((2k+1)\pi\right)^2}\cos\left((2k+1)\pi x\right)$$