solutions of homework problems
day 5.

Exercise 1

Part 1 verify that $\cos\left(\frac{n\pi x}{\ell}\right)$ $n=0,1,2,3.$ form an orthogonal set on the interval $[0,\ell]$. if $f(x)=\sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{\ell}\right)$ then find a formula for c_n

Solution $\int \cos ax \cos bx\, dx \overset{\swarrow \text{from calculus textbooks}}{=}$

$$= \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \quad \text{if } a^2 \neq b^2$$

So if $\underset{\uparrow}{n_1} \neq \underset{\uparrow}{n_2}$ then $\frac{n_1\pi}{\ell} \neq \frac{n_2\pi}{\ell}$
positive

and $\left(\cos\left(\frac{n_1\pi x}{\ell}\right), \cos\left(\frac{n_2\pi x}{\ell}\right)\right)=$

$$= \int_0^\ell \cos\left(\frac{n_1\pi x}{\ell}\right)\cos\left(\frac{n_2\pi x}{\ell}\right)dx =$$

$$= \left[\frac{\sin\left(\frac{(n_1-n_2)\pi}{\ell}x\right)}{2\left(\frac{n_1-n_2}{\ell}\pi\right)} + \frac{\sin\left(\frac{(n_1+n_2)\pi}{\ell}x\right)}{2\left(\frac{n_1+n_2}{\ell}\right)\pi}\right]_{x=0}^{x=\ell} = 0$$

If $n_1=0$ then we get

$$\int_0^\ell \underbrace{\cos\left(\frac{0\pi x}{\ell}\right)}_{=1}\cos\left(\frac{n_2\pi x}{\ell}\right)dx = \frac{\ell}{n_2\pi}\sin\left(\frac{n_2\pi x}{\ell}\right)\Big]_{x=0}^{x=\ell}$$

$$= 0$$

$$f = \sum_{n=1}^{\infty} c_n f_n \Rightarrow$$

$$(f, f_m) = \left(\sum_{n=1}^{\infty} c_n f_n, f_m\right) = c_m (f_m, f_m)$$

Thus $c_m = \dfrac{(f, f_m)}{(f_m, f_m)}$

If $m \neq 0$

$$(f_m, f_m) = \int_0^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right) dx =$$

$$2\cos^2\alpha - 1 = \cos(2\alpha) \Rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \int_0^{\ell} \frac{1 + \cos\left(\frac{2m\pi x}{\ell}\right)}{2} dx =$$

$$= \frac{x}{2} + \frac{1}{2} \frac{\ell}{2m\pi} \sin\left(\frac{2m\pi x}{\ell}\right)\Bigg]_{x=0}^{x=\ell} =$$

$$= \frac{\ell}{2}$$

Thus $\underset{m \neq 0}{c_m} = \dfrac{(f, f_m)}{\frac{\ell}{2}} = \dfrac{2}{\ell}(f, f_m) =$

$$= \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{m\pi x}{\ell}\right) dx.$$

If $m = 0$

we have $(f_0, f_0) = \int_0^{\ell} 1 \cdot 1 \, dx = \ell$

so $c_0 = \dfrac{(f, f_0)}{(f_0, f_0)} = \dfrac{1}{\ell} \displaystyle\int_0^{\ell} f(x) \cdot 1 \, dx$

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Part B Find the cosine series of $f(x) = 1-x$ on $[0,1]$

By part A if $m \neq 0$, $c_m = \frac{2}{\ell}(f, f_m) =$

$$= \frac{2}{1}\int_0^{\ell}(1-x)\underbrace{\cos\left(\frac{m\pi x}{1}\right)}_{= \left(\frac{1}{m\pi}\sin\left(\frac{m\pi x}{1}\right)\right)'}dx =$$

$$= 2\underbrace{(1-x)\frac{1}{m\pi}\sin\left(\frac{m\pi x}{1}\right)\Big]_{x=0}^{x=1}}_{=0} -$$

integration by parts

$$-2\int_0^{\ell}\frac{1}{m\pi}\sin\left(\frac{m\pi x}{1}\right)\underbrace{(-1)}_{(1-x)'}dx$$

$$= \frac{-2}{(m\pi)^2}\cos\left(\frac{m\pi x}{1}\right)\Big]_{x=0}^{x=1} =$$

$$= \frac{-2}{(m\pi)^2}\Big(\underbrace{\cos(m\pi)}_{=(-1)^m} - 1\Big) =$$

$$= \frac{-2}{(m\pi)^2}\left((-1)^m - 1\right)$$

note that if m is even then this is zero

$$c_0 = \frac{1}{\ell}(f, f_0) = \frac{1}{1}\int_0^1(1-x)\,1\,dx = \frac{1}{2}$$

Thus the answer is the following cosine series

$$\frac{1}{2}\cdot \underbrace{1}_{=\cos\left(\frac{0\pi x}{1}\right)} + \sum_{m=1}^{\infty} \frac{-2}{(m\pi)^2}\left((-1)^m-1\right)\cos\left(\frac{m\pi x}{1}\right)$$

This answer is <u>true</u> but if desired one can observe that $(-1)^m-1=0$ if m is even while if m is odd we get -2. m is encoded by $(2k+1)$ thus in principle one can write this as

$$\frac{1}{2}\cdot 1+\sum_{k=0}^{\infty}\frac{(-2)}{\left((2k+1)\pi\right)^2}\cos\left(\frac{(2k+1)\pi x}{2}\right)$$

if you start with k=1 the answer would be wrong

Problem 3 page 214

$f, g \in L^2$ prove the Cauchy-Schwarz inequality $|(f,g)| \le \|f\|\,\|g\|$

Hint define $\varphi(t) = (f+tg, f+tg)$ and explore it further

$\|f+tg\| \ge 0$ $\quad \mathbb{R}$

$$\varphi(t) = (f+tg, f+tg) = (f,f) + 2(f,tg) + (tg,tg) = \|f\|^2 + 2t(f,g) + t^2\|g\|^2$$

Thus $\|f\|^2 + 2t(f,g) + t^2\|g\|^2 \ge 0$ for all t. Thus the quadratic equation

$$\underbrace{\|g\|^2}_{a} t^2 + \underbrace{2(f,g)}_{b} t + \underbrace{\|f\|^2}_{c} = 0$$

has no roots and so

$\underline{b^2 - 4ac} \le 0$

discriminant of the quadratic equation.

So $\quad (2(f,g))^2 - 4\|g\|^2\|f\|^2 \le 0$

$$(f,g)^2 \le \|f\|^2\|g\|^2$$

$$|(f,g)| \le \|f\|\,\|g\|$$