Math 46 sketches of solutions of homework problems day 4.

Exercise 1. c page 40

$u' - tu = t^2 u^2$ Put $w = u^{1-2} = u^{-1}$

$\frac{1}{u^2} u' - t \frac{1}{u} = t^2$ $w' = -\frac{1}{u^2} u'$

$-w' - tw = t^2$

$w' + \underbrace{t}_{p} w = \underbrace{-t^2}_{q}$ $\Big| \times$ multiply by $e^{\int t dt} = e^{\frac{t^2}{2}}$

$\underbrace{w' e^{t^2/2} + t w e^{t^2/2}}_{\parallel \; \left(w e^{t^2/2}\right)'} = -t^2 e^{\frac{t^2}{2}}$

$\Rightarrow \quad w e^{t^2/2} = -\int t^2 e^{\frac{t^2}{2}} dt + c$

$w = -e^{-\frac{t^2}{2}} \int t^2 e^{\frac{t^2}{2}} dt + c e^{-\frac{t^2}{2}}$

$u = \frac{1}{w}$ ☺

It is important to remember that the real answer is u rather than w.

Note that when dividing by u we missed a solution $u(t) \equiv 0$ ☺ that should also be counted

Exercise 1.d

$t^2u'' - 3tu' + 4u = 0$

Cauchy Euler equation

$u(t) = t^m$

$t^2 m(m-1)t^{m-2} - 3tmt^{m-1} + 4t^m = 0$

$t^m(m(m-1) - 3m + 4) = 0$

$m^2 - 4m + 4 = 0$

$m_{1,2} = \frac{4 \pm \sqrt{16-16}}{2} = 2$

coinciding roots so

$u_1(t) = t^2 \qquad u_2(t) = t^2 \ln t$

So the general solution is

$u(t) = c_1 t^2 + c_2 t^2 \ln t$

Exercise 1.e

$u'' + 9u = 3\sec(3t)$

We first solve the homogeneous problem

$u'' + 9u = 0$

The characteristic equation is

$r^2 + 9 = 0 \qquad r_{1,2} = \pm 3i$

$\tilde{u}_1(t) = e^{3i} \qquad \tilde{u}_2(t) = e^{-3i} \qquad u_1(t) = \cos 3t$

$u_2(t) = \sin 3t$

Thus the fundamental solution of the homogeneous problem is $u(t) = c_1 e^{3t} + c_2 e^{-3t}$

We have to find a particular solution

$$u_p(t) = \int_0^t \frac{u_1(s)u_2(t) - u_2(s)u_1(t)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} f(s)\,ds$$

$3\sec 3t$

$$u_1(s)u_2'(s) - u_2(s)u_1'(s) = (\cos 3t)(3)(\cos 3t) - \sin(3t)\cdot(-3)\sin 3t = 3.$$

$$u_p(t) = \sin(3t)\int_0^t \frac{3\cos(3s)}{\cos(3s)}ds - \cos(3t)\int_0^t \frac{3\sin(3s)}{\cos(3s)}ds$$

$$= \sin(3t)\,3t + \cos 3t \ln(|\cos 3s|)\Big]_{s=0}^{s=t} =$$

$$= \sin(3t) + \cos(3t)\ln|\cos 3t|$$

Exercise 1.f

$$u'' + tu'^2 = 0$$

Put $y = u'$ we get

$$y' + ty^2 = 0$$

$$\frac{dy}{dt} = -ty^2$$

$$\int \frac{1}{y^2}dy = \int -t\,dt$$

$$-\frac{1}{y} = -\frac{t^2}{2} + c$$

$$\frac{1}{y} = \frac{t^2}{2} + c = \frac{t^2 + \overbrace{2c}^{=\tilde{c}}}{2}$$

$$y = \frac{2}{t^2 + \tilde{c}}$$

$$u' = y = \frac{2}{t^2 + \tilde{c}}$$

$$u(t) = \int \frac{2}{t^2 + \tilde{c}}\, dt + D$$

note that there should be two constants in your general solution

Exercise 1 i

$$2t^2 u'' + 3tu' - u = 0$$

$$u(t) = t^m$$

$$2t^2 m(m-1)t^{m-2} + 3t(m\ )t^{m-1} - t^m = 0$$

$$\left(2m(m-1) + 3(m-1) - 1\right) t^m = 0$$

$$2m^2 + m - 1 = 0$$

$$m_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$m_1 = \frac{1}{2} \quad m_2 = -1$$

$$u(t) = c_1 t^{\frac{1}{2}} + c_2 t^{-1}$$

j) **Exercise 1j**

$u' - 2tu = 1$

$$\mu(t) = e^{\int -2t\,dt} = e^{-t^2}$$

$$u' + \underbrace{(-2t)}_{p(t)}u = \underbrace{1}_{q(t)}$$

$$\underbrace{(u' + (-2t)u)\, e^{-t^2}}_{\overset{\parallel}{(ue^{-t^2})'}} = 1e^{-t^2}$$

Thus $(ue^{-t^2})' = e^{-t^2}$

$$\Rightarrow \quad ue^{-t^2} = \int e^{-t^2}\,dt + c$$

$$u(t) = e^{t^2}\int e^{-t^2}\,dt + ce^{t^2}$$

Exercise 1n

$$u'' + \omega^2 u = \cos\omega t$$

We first solve the homogeneous equation $u'' + \omega^2 u = 0$ (*)

The characteristic equation is

$$r^2 + \omega^2 = 0 \qquad r_{1,2} = \pm i\omega$$

Thus the fundamental solution of (*) is $u(t) = c_1\underbrace{\cos\omega t}_{u_1(t)} + c_2\underbrace{\sin\omega t}_{u_2(t)}$

Now we search for a particular solution of $u'' + \omega^2 u = \cos\omega t$

$$u_p(t) = \int_0^t \frac{u_1(s)u_2(t) - u_2(s)u_1(t)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} f(s)\,ds$$

$$u_1(s)u_2'(s) - u_2(s)u_1'(s) =$$

$$= \cos(\omega s)\underbrace{\omega\cos(\omega s)}_{u_2'(s)} - \sin(\omega s)\,\omega(-\sin(\omega s))$$

$$= \omega$$

$$u_p(t) = \int_0^t \frac{\cos(\omega s)\sin(\omega t) - \sin(\omega s)\cos(\omega t)}{\omega} \cdot \cos(\omega s)\,ds =$$

$$= \int_0^t \frac{\cos^2(\omega s)}{\omega}\,ds\,\sin(\omega t) -$$

$$- \int_0^t \frac{\sin(\omega s)\cos(\omega s)}{\omega}\,ds\,\cos(\omega t) =$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \int_0^t \frac{1+\cos(2\omega s)}{2\omega}\,ds\,\sin(\omega t) -$$

$$-\left(\frac{1}{2}\,\frac{\sin^2(\omega s)}{\omega^2}\Big]_{s=0}^{s=t}\right)\cos\omega t$$

$$= \left(\frac{1}{2\omega} s + \frac{\sin(2\omega s)}{(2\omega)^2}\right)\Big]_{s=0}^{s=t} \sin\omega t -$$

$$- \left(\frac{1}{2} \frac{\sin^2(\omega s)}{\omega^2}\Big]_{s=0}^{s=t}\right) \cos\omega t =$$

$$= \left(\frac{1}{2\omega} t + \sin(2\omega t)\right) \sin\omega t -$$

$$- \frac{1}{2} \frac{\sin^2(\omega t)}{\omega^2} \cos\omega t$$

The general solution is

$u(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + u_p(t)$

that we found above.