Solutions for Day 2 homework

Exercise 3 page 30



$$m' = ax^2 - bx^3$$

$m$ is measured in $M$

$m'$ — " — in $\frac{M}{T}$

$a$ in $\frac{M}{TL^2}$ so that $ax^2$ is in $\frac{M}{T}$

$b$ in $\frac{M}{TL^3}$ so that $bx^3$ is in $\frac{M}{T}$

$m = \rho x^3$ so $\rho$ is in $\frac{M}{L^3}$

$$\underbrace{\frac{d}{dt}(\rho x^3)}_{m'} = ax^2 - bx^3$$

$$3\rho x^2 x' = ax^2 - bx^3$$

(*) $3\rho x' = a - bx$ ← initially grows fast but then slows down since volume and hence usage of nutrients is big

$x(0) = 0$
initial condition

$\bar{t} = \frac{t}{\left(\frac{\rho}{b}\right)}$ ← measured in $T$

$\bar{x} = \frac{x}{\left(\frac{a}{b}\right)}$ ← measured in $L$

$$\frac{dx}{dt} = \frac{d\left(\frac{a}{b}\bar{x}\right)}{dt} = \frac{a}{b}\frac{d\bar{x}}{dt} = \frac{a}{b}\frac{d\bar{x}}{d\bar{t}}\underbrace{\frac{d\bar{t}}{dt}}_{\frac{b}{\rho}} = \frac{a}{\rho}\frac{d\bar{x}}{d\bar{t}}$$

So (*) becomes

$$3\cancel{\beta}\,\frac{a}{\cancel{\beta}}\,\frac{d\bar{x}}{d\bar{t}} = a - \cancel{b}\left(\frac{a}{\cancel{b}}\right)\bar{x}$$

$$3\,\frac{d\bar{x}}{d\bar{t}} = 1 - \bar{x}$$

$$\int\frac{d\bar{x}}{1-\bar{x}} = \int\frac{1}{3}\,d\bar{t}$$

$x(0)=0$ gives $\bar{x}(0)=0$
so for small times $1-\bar{x} \geq 0$

$$-3\ln(1-\bar{x}) = \bar{t} + c$$

$$\ln(1-\bar{x}) = -\frac{\bar{t}}{3} + \tilde{c}$$

$$e^{\ln(1-\bar{x})} = e^{-\bar{t}/3}\,\underbrace{e^{\tilde{c}}}_{=D}$$

$$1-\bar{x} = De^{-\bar{t}/3} \Rightarrow \bar{x}(\bar{t}) = 1 - De^{-\frac{\bar{t}}{3}}$$

$x(0)=0 \Rightarrow D=1$

$$\bar{x}(\bar{t}) = 1 - 1e^{-\frac{\bar{t}}{3}}$$

Now we return back to the initial variables

$$\frac{x}{\left(\frac{a}{b}\right)}(t) = 1 - e^{-\frac{bt}{3\beta}}$$

$$x(t) = \frac{a}{b} - \frac{a}{b}e^{-\frac{bt}{3\beta}} =$$

$$= \frac{a}{b}\left(1 - e^{-\frac{bt}{3\beta}}\right)$$

in the limit $x(t) = \frac{a}{b}$ so it consumes exactly as much as it eats $a\left(\frac{a}{b}\right)^2 = b\left(\frac{a}{b}\right)^3$

Exercise 12 page 33

$F(x,t) = -kxe^{-t/a}$

measurements

$F - \frac{M \cdot L}{T^2}$ $\quad a - T$ so that $\frac{t}{a}$ is dimensionless

$k - \frac{M}{T^2}$

The equation we get is

$m x'' = -kxe^{-t/a}$

$x(0) = \ell$

$x'(0) = V$ $\qquad \bar{x} = \frac{x}{\ell} \Rightarrow x = \ell \bar{x}$

$\bar{t} = \frac{t}{a}$

$$\frac{dx}{dt} = \frac{d(\ell \bar{x})}{dt} = \ell \frac{d\bar{x}}{dt} = \ell \frac{d\bar{x}}{d\bar{t}} \frac{d\bar{t}}{dt} = \ell \frac{d\bar{x}}{d\bar{t}} \frac{1}{a}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{\ell}{a}\frac{d\bar{x}}{d\bar{t}}\right) = \frac{\ell}{a}\frac{d}{d\bar{t}}\left(\frac{d\bar{x}}{d\bar{t}}\right)\underbrace{\frac{d\bar{t}}{dt}}_{=\frac{1}{a}} = \frac{\ell}{a^2}\frac{d^2\bar{x}}{d\bar{t}^2}$$

so we get

$$\frac{m\ell}{a^2}\frac{d^2\bar{x}}{d\bar{t}^2} = -k(\ell\bar{x})e^{-\bar{t}} \qquad \frac{d\bar{x}}{d\bar{t}}(0) = \frac{aV}{\ell}$$

$x(0) = \ell \Rightarrow \bar{x}(0) = 1$

$\frac{dx}{dt}(0) = V \Rightarrow \frac{dx}{dt}(0) = \frac{\ell}{a}\frac{d\bar{x}}{d\bar{t}}(0) = V \Rightarrow$

We get

$$\frac{m\ell}{a^2}\frac{d^2\bar{x}}{d\bar{t}^2} = -k(\ell\bar{x})e^{-\bar{t}}$$

$$\bar{x}(0) = 1$$

$$\frac{d\bar{x}}{d\bar{t}}(0) = \frac{av}{\ell}$$

$\Rightarrow$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\underbrace{\frac{ka^2}{m}}_{\overset{def}{=}\varepsilon}\bar{x}e^{-\bar{t}}$$

$$\bar{x}(0) = 1$$

$$\frac{d\bar{x}}{d\bar{t}}(0) = \underbrace{\frac{av}{\ell}}_{\overset{def}{=}\mu}$$

$$[\varepsilon] = \frac{M^1}{T^2}\frac{T^2}{M^1} = 1$$

$\uparrow$ dimensionless

$$[\mu] = \frac{T\frac{L}{T}}{L} = 1$$

$\uparrow$ also dimensionless

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\varepsilon\bar{x}e^{-\bar{t}}$$

$$\bar{x}(0) = 1$$

$$\frac{d\bar{x}}{d\bar{t}}(0) = \mu$$

is the answer

Note that it is important to bring all the coefficients to the dimensionless form