Math 46

page 1

Solutions of homework problems

Day 29

Exercise 1 A

Find $\left(\mathcal{F}(\delta(x-x_0))\right)''$

$$\left[\left(\mathcal{F}(\delta(x-x_0))\right)'',\ \underset{\substack{\uparrow\\ \text{test functions}\\ \text{from } S}}{\varphi(x)}\right] =$$

$$= \left[\mathcal{F}(\delta(x-x_0)),\ (-1)^2\varphi''(x)\right] =$$

$$= 2\pi\left[\delta(x-x_0),\ \mathcal{F}^{-1}(\varphi''(x))\right] =$$

$$= \cancel{2\pi}\left[\delta(x-x_0)\ \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\varphi''(\xi)\,e^{-i\xi x}\,d\xi\right]$$

$$= \int_{-\infty}^{\infty}\varphi''(\xi)\,\underbrace{e}\,^{-i\xi x_0}\,d\xi =$$

$$\underbrace{\varphi'(\xi)\,e^{-i\xi x_0}\Big]_{\xi=-\infty}^{\xi=\infty}}_{0 \text{ since } \varphi\in S} - \int_{-\infty}^{\infty}\varphi'(\xi)(-ix_0)\,e^{-i\xi x_0}\,d\xi$$

$$= \underbrace{ix_0\,e^{-i\xi x_0}\,\varphi(\xi)\Big]_{\xi=-\infty}^{\xi=\infty}}_{0 \text{ since } \varphi\in S} - \int_{-\infty}^{\infty}(ix_0)^2\varphi(\xi)\,e^{-i\xi x_0}\,d\xi$$

page 2

$$= \underbrace{(i x_0)^2}_{= -x_0^2} \int_{-\infty}^{\infty} \varphi(\xi) e^{-i\xi x_0} d\xi =$$

$$= \left(-x_0^2 e^{-i\xi x_0}, \varphi(\xi)\right) \Rightarrow$$

$$\left(\mathcal{F}(\delta(x-x_0))\right)'' = -x_0^2 e^{-i\xi x_0}$$

(B) $\left(\mathcal{F}(\delta''(x-x_0)), \varphi(x)\right) =$

$$= 2\pi \left(\delta''(x-x_0), \mathcal{F}^{-1}(\varphi)\right) =$$

$$= \cancel{2\pi} \left(\delta''(x-x_0), \cancel{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \varphi(\xi) d\xi\right)$$

$$= \left(\delta(x-x_0), (-1)^2 \frac{d^2}{dx^2} \int_{-\infty}^{\infty} e^{-i\xi x} \varphi(\xi) d\xi\right)$$

$$= \left(\delta(x-x_0), \int_{-\infty}^{\infty} (-i\xi)^2 e^{-i\xi x} \varphi(\xi) d\xi\right)$$

$$= \int_{-\infty}^{\infty} (-i\xi)^2 e^{-i\xi x_0} \varphi(\xi) d\xi =$$

$$= \left((-i\xi)^2 e^{-i\xi x_0}, \varphi(\xi)\right) =$$

$$= \left(-\xi^2 e^{-i\xi x_0}, \varphi(\xi)\right) \Rightarrow$$

$$\mathcal{F}(\delta''(x-x_0)) = -\xi^2 e^{-i\xi x_0}$$

Exercise 2

$$y^{(4)} - 4y = 0 \qquad y(0)=1 \quad y'(0)=0$$
$$y''(0)=-2 \quad y'''(0)=0$$

$$\mathcal{L}(y^{(4)} - 4y) = \mathcal{L}(0) = 0$$

$$s^4 \mathcal{L}(y) - s^3 \overset{=1}{y(0)} - s^2 \overset{=0}{y'(0)} - s\overset{=-2}{y''(0)} - \underset{=0}{y'''(0)}$$
$$-4\mathcal{L}(y) = 0$$

$$(s^4-4)\mathcal{L}(y) - s^3 \cdot 1 + 2s = 0$$

$$\mathcal{L}(y) = \frac{-2s+s^3}{s^4-4} = \frac{s^3-2s}{(s^2+2)(s^2-2)} =$$

$$= \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2-2}$$

Bring to the common denominator

$$As^3 - 2As + Bs^2 - 2B + Cs^3 + 2Cs + Ds^2 + 2D$$
$$= s^3 - 2s$$

$$(A+C)s^3 + (B+D)s^2 + (2C-2A)s + 2(D-B) =$$
$$= 1s^3 + 0s^2 - 2s + 01$$

$$A+C=1 \quad (1)$$
$$B+D=0 \quad (2)$$
$$2C-2A=-2 \quad (3)$$
$$D-B=0 \quad (4)$$

$(2)+(4) \Rightarrow B=D=0$

$$A+C=1$$
$$C-A=-1$$
$$2C=0 \qquad A=1$$

$$= \frac{s}{s^2+2}$$

$$y = \mathcal{L}^{-1}\left(\frac{s}{s^2+2}\right) = \cos\sqrt{2}\,t$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \qquad a = \sqrt{2}$$

$$y(t) = \cos\sqrt{2}\,t$$

Exercise 1 page 415 (pages)

① Is the function e^{x^2} locally integrable on $\mathbb{R}$.

Yes since every closed bounded $K \subset \mathbb{R}$ is contained within some $[c,d] \subset \mathbb{R}$

$$\int_K |e^{x^2}|\,dx \leq \int_c^d |e^{x^2}|\,dx = \int_c^d e^{x^2}\,dx \leq$$

Now e^{x^2} is continuous on $[c,d]$ so it achieves its maximal value say M

$$\leq M\cdot(d-c) < \infty$$

② Does it generate a distribution in $D'(\mathbb{R})$. Yes, since it is locally integrable.

③ Does it generate a temepered distribution? No. Since

$$\varphi(x) = e^{-x^2} \in S$$

$$\int_{-\infty}^{\infty} e^{x^2} e^{-x^2}\,dx = \int_{-\infty}^{\infty} 1\,dx = \infty$$

Exercise 2 page 415

Show that for any locally integrable f on $\mathbb{R}$, the function $u(x,y)=f(x-y)$ is a weak solution to the equation $u_x+u_y=0$ on $\mathbb{R}^2$.

Solution It suffices to verify that

$$\left(\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}u,\varphi\right)\overset{?}{=}(0,\varphi)=0$$

$$\shortparallel$$

$$\left(u,-\frac{\partial\varphi}{\partial x}\right)+\left(u,-\frac{\partial\varphi}{\partial y}\right)=$$

$$=\int_{\mathbb{R}^2} f(x-y)\left(-\frac{\partial\varphi}{\partial x}\right)dx\,dy+\int_{\mathbb{R}^2} f(x-y)\left(-\frac{\partial\varphi}{\partial y}\right)dx\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\left(-\frac{\partial\varphi}{\partial x}\right)+\left(-\frac{\partial\varphi}{\partial y}\right)dx\,dy=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\left(-\frac{\partial\varphi}{\partial x}\right)dx\,dy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\left(-\frac{\partial\varphi}{\partial y}\right)dy\,dx$$

$$=\int_{-\infty}^{\infty}\underbrace{\left[-\varphi(x,y)f(x-y)\right]_{x=-\infty}^{x=\infty}}_{\substack{\shortparallel\\ 0\\ \text{since }\varphi\in S}}+\int_{-\infty}^{\infty}\frac{df}{dt}(x-y)\cdot\varphi\,dx\,dy$$

↑ by parts

evaluated at

page 7

$$+\int_{-\infty}^{\infty} \underbrace{-\varphi(x,y)\, f(x-y)\Big]_{y=-\infty}^{y=+\infty}}_{=0 \text{ since } \varphi\in S} + \int_{-\infty}^{\infty} \frac{df}{dt}(x-y)\,(-1)\,\underset{\uparrow\ \varphi(x,y)}{}dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,y)\frac{df}{dt}(x-y)\,1\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{df}{dt}(x-y)(-1)\,\varphi(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x,y)\frac{df}{dt}(x-y)\,(1+(-1))\,dx\,dy = 0$$

$\uparrow$
Fubini's Theorem