Math 46 Solutions to homework problems Day 28

Exercise 8 page 397

$u_t = u_{xx} + F(x,t), \quad x \in \mathbb{R} \quad t > 0$

$u(x,0) = 0 \qquad x \in \mathbb{R}$

Solution apply the Fourier transform

$$\mathcal{F}(u_t) = \mathcal{F}(u_{xx}) + \mathcal{F}(F(x,t))$$

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) = (-i\xi)^2 \hat{u}(\xi,t) + \hat{F}(\xi,t)$$

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) + \xi^2 \hat{u}(\xi,t) = \hat{F}(\xi,t)$$

if you freeze ξ, then this is a linear equation so we multiply both sides by $e^{\xi^2 t}$

$$\underbrace{\frac{\partial}{\partial t}\hat{u}(\xi,t)e^{\xi^2 t} + \xi^2\hat{u}(\xi,t)e^{\xi^2 t}}_{\displaystyle \frac{\partial}{\partial t}\left(\hat{u}(\xi,t)e^{\xi^2 t}\right)} = \hat{F}(\xi,t)e^{\xi^2 t}$$

$$\Rightarrow \hat{u}(\xi,t)e^{\xi^2 t} = \int_0^t \hat{F}(\xi,t)e^{\xi^2 t}\,dt + C(\xi)$$

$$u(\xi,0) = \hat{0} = 0$$

$$\parallel$$

$$\underbrace{\int_0^0 \hat{F}(\xi,t)\, e^{\xi^2 t}\, dt}_{=0} + c(\xi) \Rightarrow c(\xi) = 0$$

$$\Rightarrow \hat{u}(\xi,t)\, e^{\xi^2 t} = \int_0^t \hat{F}(\xi,t)\, e^{\xi^2 t}\, dt$$

$$\Rightarrow \hat{u}(\xi,t) = e^{-\xi^2 t} \int_0^t \hat{F}(\xi,t)\, e^{\xi^2 t}\, dt$$

$$u(x,t) = \mathcal{F}^{-1}\left(e^{-\xi^2 t} \int_0^t \hat{F}(\xi,t)\, e^{\xi^2 t}\, dt \right)$$

Exercise 9.b page 397

Find the Fourier transform of the n-dimensional Gaussian $u(\vec{x}) = e^{-a|\vec{x}|^2}$

Solution

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-a|\vec{x}|^2} e^{i\vec{\xi}\cdot\vec{x}}\, d\vec{x} =$$

$$= \int_{\mathbb{R}^n} e^{-a\left(\sum_{j=1}^n x_j^2\right)} e^{i\sum_{j=1}^n \xi_j x_j}\, d\vec{x} =$$

$$= \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n\text{-integrals}} e^{-a\left(\sum_{j=1}^n x_j^2\right)} e^{i\sum_{j=1}^n \xi_j x_j}\, dx_1 \ldots dx_n$$

$$\underset{\substack{\uparrow \\ \text{Fubini's} \\ \text{Theorem}}}{=} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{(n-1)\text{-integrals}} \overbrace{\int_{-\infty}^{\infty} e^{-ax_n^2} e^{i\xi_n x_n}\, dx_n}^{\text{Fourier transform along } x_n \; \mathcal{F}_{x_n}(e^{-ax_n^2})}$$

$$\cdot\, e^{-a\sum_{j=1}^{n-1} x_j^2} e^{i\sum_{j=1}^{n-1} \xi_j x_j}\, dx_1, \ldots, dx_{n-1}$$

$$= \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{(n-1)\text{-integrals}} \sqrt{\frac{\pi}{a}}\, e^{-\xi_n^2/4a}\,\cdot$$

$$e^{-a\sum_{j=1}^{n-1} x_j^2} \cdot e^{i\sum_{j=1}^{n-1} \xi_j x_j}\, dx_1 \ldots dx_{n-1}$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-a\sum_{j=1}^{n-1} x_j^2} e^{i\sum_{j=1}^{n-1} x_j \xi_j}\, dx_1 \dots dx_{n-1}$$

$$= \left(\sqrt{\frac{\pi}{a}}\right)^n e^{-\xi_n^2/4a}\, e^{-\xi_{n-1}^2/4a} \cdots e^{-\xi_1^2/4a} =$$

$$= \left(\sqrt{\frac{\pi}{a}}\right)^n e^{-\frac{1}{4a}\sum_{j=1}^{n} \xi_j^2} = \left(\sqrt{\frac{\pi}{a}}\right)^n e^{-\frac{1}{4a}|\xi|^2}$$

Exercise 9.c page 397

Solve the Cauchy problem for the diffusion equation

$$u_t = D\Delta u \qquad \vec{x} \in \mathbb{R}^n,\ t>0$$

$$u(\vec{x},0) = f(\vec{x}) \qquad \vec{x} \in \mathbb{R}^n$$

Solution

Apply the n-dimensional Fourier transform

$$\frac{\partial}{\partial t}\hat{u}(\vec{\xi},t) = \mathcal{F}(D\Delta u) = D(-|\vec{\xi}|^2)\hat{u}(\vec{\xi})$$

$$\Rightarrow \hat{u}(\vec{\xi},t) = c(\vec{\xi})\, e^{-D|\vec{\xi}|^2 t}$$

$$\hat{u}(\vec{\xi},0) = c(\vec{\xi})\underbrace{e^{-D|\vec{\xi}|^2\cdot 0}}_{=1} = c(\vec{\xi})$$

$$\parallel$$

$$\int_{\mathbb{R}^n} \underbrace{u(\vec{x},0)}_{= f(\vec{x})} e^{i\vec{\xi}\cdot\vec{x}}\, d\vec{x} = \hat{f}(\vec{\xi})$$

$$\Rightarrow \hat{u}(\xi,t) = \hat{f}(\vec{\xi})\, e^{-D|\vec{\xi}|^2 t}$$

$$\Rightarrow u(\vec{x},t) = \mathcal{F}^{-1}\left(\hat{f}(\vec{\xi})\, e^{-D|\vec{\xi}|^2 t}\right)$$

Convolution $\quad u * v(\vec{x}) \overset{def}{=} \int_{\mathbb{R}^n} u(\vec{x}-\vec{y})\, v(\vec{y})\, dy$

$$\mathcal{F}(u * v)(\xi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(\vec{x}-\vec{y})\, v(\vec{y})\, d\vec{y}\; e^{i\vec{\xi}\cdot\vec{x}}\, d\vec{x} =$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(\vec{x}-\vec{y})\, v(\vec{y})\, e^{i\vec{\xi}\cdot\vec{x}}\, d\vec{x}\, d\vec{y} =$$

↑ Fubini's Theorem $\qquad$ Put $\vec{r} = \vec{x} - \vec{y}$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(\vec{r})\, v(\vec{y})\, e^{i\vec{\xi}\cdot(\vec{r}+\vec{y})}\, d\vec{r}\, d\vec{y} =$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(\vec{r})\, v(\vec{y})\, e^{i\vec{\xi}\cdot\vec{r}}\, e^{i\vec{\xi}\cdot\vec{y}}\, d\vec{r}\, d\vec{y} =$$

$$= \int_{\mathbb{R}^n} u(\vec{r})\, e^{i\vec{\xi}\cdot\vec{r}}\, dr \int_{\mathbb{R}^n} v(y)\, e^{i\vec{\xi}\cdot y}\, dy =$$

$$= \hat{u}(\vec{\xi})\, \hat{v}(\xi)$$

$$\Rightarrow u(\vec{x}, t) = \underbrace{\mathcal{F}^{-1}(\hat{f}(\xi))}_{= f(\vec{x})} * \mathcal{F}^{-1}\left(e^{-D|\vec{\xi}|^2 t}\right)$$

$$\mathcal{F}\left(e^{-a|\vec{x}|^2}\right) = \left(\sqrt{\frac{\pi}{a}}\right)^n e^{-\frac{1}{4a}|\vec{\xi}|^2}$$

$$e^{-D|\vec{\xi}|^2 t} \qquad \left(\frac{1}{\sqrt{\pi 4Dt}}\right)^n \left(\sqrt{\frac{\pi}{\left(\frac{1}{4Dt}\right)}}\right)^n e^{-\frac{|\vec{\xi}|^2}{4\left(\underset{\uparrow a}{\frac{1}{4Dt}}\right)}}$$

$$\Rightarrow \mathcal{F}^{-1}\left(e^{-D|\vec{\xi}|^2 t}\right) = \left(\frac{1}{\sqrt{4\pi D t}}\right)^n e^{-\frac{1}{4Dt}|\vec{x}|^2}$$

$$\Rightarrow u(\vec{x},t) = f(\vec{x}) * \left(\left(\frac{1}{\sqrt{4\pi D t}}\right)^n e^{-\frac{1}{4Dt}|\vec{x}|^2}\right)$$

$$= \int_{\mathbb{R}^n} f(\vec{y}) \left(\frac{1}{\sqrt{4\pi D t}}\right)^n e^{-\frac{1}{4Dt}|\vec{x}-\vec{y}|^2} dy$$