Math 46

Solutions to homework problems

Day 17

Exercise 7 page 396

Use Fourier transforms to find the solution to the advection diffusion equation

$$u_t - c u_x - u_{xx} = 0 \qquad x \in \mathbb{R} \quad t > 0$$
$$u(x,0) = f(x) \qquad x \in \mathbb{R} \quad u(x,t)$$

Apply the Fourier transform along x

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) - c(-i\xi)\hat{u}(\xi,t) - (-i\xi)^2\hat{u}(\xi,t) = 0$$

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) + (ci\xi + \xi^2)\hat{u}(\xi,t) = 0$$

$$\Rightarrow \hat{u}(\xi,t) = D(\xi)\, e^{-(ci\xi+\xi^2)t}$$

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$

$$\hat{u}(\xi,0) = D(\xi)\underbrace{e^{-(ci\xi+\xi^2)\cdot 0}}_{=1} \Rightarrow D(\xi) = \hat{f}(\xi)$$

Thus $\hat{u}(\xi, t) = \hat{f}(\xi)\, e^{-(ci\xi + \xi^2)t}$ (page 2)

$$e^{(ci\xi + \xi^2)t} = e^{-((\xi + \frac{ci}{2})^2 + \frac{c^2}{4})t} =$$

$$= e^{-(\xi + \frac{ci}{2})^2 t}\, e^{-\frac{c^2 t}{4}}.$$

Thus $u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(\xi)\, e^{-(\xi + \frac{ci}{2})^2 t}\right)$

because $e^{(c^2t)/4}$ does not involve ξ

$$= e^{-\frac{c^2 t}{4}}\, \mathcal{F}^{-1}\left(\hat{f}(\xi)\, e^{-(\xi + \frac{ci}{2})^2}\right) =$$

$$= e^{-\frac{c^2 t}{4}}\, f(x) * \mathcal{F}^{-1}\left(e^{-(\xi + \frac{ci}{2})^2 t}\right)$$

convolution

Now we compute $\mathcal{F}^{-1}\left(e^{-(\xi + ci/2)^2 t}\right)$

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}$$

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{1}{4t}x^2}\right) = \left(\frac{1}{\sqrt{4\pi t}}\right) \underbrace{\sqrt{4\pi t}}_{= \sqrt{\frac{\pi}{a}}}\, \underbrace{e^{-\xi^2/4(\frac{1}{4t})}}_{= e^{-\xi^2 t}}$$

use it with $a = \frac{1}{4t}$

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{1}{4t}x^2}\right) = e^{-\xi^2 t}$$

As we know from your homework exercise 5.b (page 3)

$$\mathcal{F}(e^{iax}u)(\cancel{x}) = \hat{u}(\xi + a)$$

Thus

$$\mathcal{F}\left(e^{i\left(\frac{ci}{2}\right) x}\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{1}{4t}x^2}\right) = \quad \frac{ci}{2} \leftarrow a$$

$$= e^{-\left(\xi + \frac{ci}{2}\right)^2 t}$$

$$\Rightarrow \mathcal{F}^{-1}\left(e^{-\left(\xi+\frac{ci}{2}\right)t}\right) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{c}{2}x}\, e^{-\frac{1}{4t}x^2}$$

Thus

$$u(x,t) = e^{-\frac{c^2 t}{4}} f(x) * \mathcal{F}^{-1}\left(e^{-\left(\xi+\frac{ci}{2}\right)^2 t}\right)$$

$$= \int_{-\infty}^{\infty} e^{-\frac{c^2 t}{4}} f(x-y)\, \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{c}{2}y}\, e^{-\frac{1}{4t}y^2}\, dy$$

$$= \frac{e^{-\frac{c^2 t}{4}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x-y)\, e^{-\frac{c}{2}y - \frac{1}{4t}y^2}\, dy$$

Exercise 11 page 397

Use the Plancherel relation to evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi$$

$$\mathcal{F}\left(e^{-1|x|}\right) = \frac{2 \cdot 1}{1+\xi^2}$$

$$\mathcal{F}\Big(\underbrace{\tfrac{1}{2} e^{-1|x|}}_{u(x)}\Big) = \frac{1}{2} \cdot \frac{2}{1+\xi^2} = \underbrace{\frac{1}{1+\xi^2}}_{= \hat{u}(\xi)}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^2} = \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi$$

$$= 2\pi \int_{-\infty}^{\infty} |u(x)|^2 dx =$$

$$= \cancel{2}\pi \int_{-\infty}^{\infty} \underbrace{\frac{1}{\cancel{2}} e^{-|x|}}_{\text{even function}} dx =$$

$$= 2 \cdot \pi \int_{0}^{\infty} e^{-|x|} dx = 2\pi \int_{0}^{\infty} e^{-x} dx$$

$$= 2\pi \lim_{R \to \infty} \int_{0}^{R} e^{-x} dx =$$

$$= \lim_{R \to \infty} 2\pi \left(-e^{-x} \Big]_{x=0}^{x=R} \right) = \lim_{R \to \infty} 2\pi \left(-e^{-R} + e^{0}\right) = 2\pi$$

Exercise 13 page 397

Solve the initial value problem

(*) $u_t = D u_{xx}$ $x > 0$ $t > 0$

$u(x,0) = f(x)$ $x > 0$

$u(0,t) = 0$ $t > 0$ ← needed in order to have a continuous odd extension

By extending f to $\mathbb{R}$ as the odd function and then using the Fourier transform.

Put $\tilde{f}$ to be the odd extension

Solution Assume $u(x,t)$ is a solution.

Put $\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } x \geq 0,\ t > 0 \\ -u(-x,t) & \text{if } x < 0\ t > 0 \end{cases}$

Then $\tilde{u}(0,t) = 0 \quad \forall t$

$\underset{\substack{\uparrow \\ x\text{-negative}}}{\tilde{u}(x,0)} = -u(\underset{\substack{\uparrow \\ \geq 0}}{-x},0) = -f(-x)$

$= \tilde{f}_{odd}(x)$

↑ odd extension of f

Now $\tilde{u}$ satisfies in the first quadrant Let us check

that $\tilde{u}$ satisfies (*)
in the second quadrant

$$\frac{\partial}{\partial t}\tilde{u}(x,t) \overset{?}{=} D\frac{\partial^2}{\partial x^2}\tilde{u}(x,t)$$

$$\frac{\partial}{\partial t}\tilde{u}(x,t) = \frac{\partial}{\partial t}\left(-u(-x,t)\right) \qquad D\frac{\partial^2}{\partial x^2}\tilde{u}(x,t) = D\frac{\partial^2}{\partial x^2}\left(-u(-x,t)\right)$$

$$\frac{\partial}{\partial t}\left(-u(-x,t)\right) = -\frac{\partial u}{\partial t}\Big|_{(\underset{\geq 0}{-x},t)} = -(-1)^2 D\frac{\partial^2}{\partial u}\Big|_{(-x,t)}$$

computed at

is indeed true.

Thus $\tilde{u}(x,t)$ satisfies

$$\left.\begin{aligned} &\tilde{u}_t = D\tilde{u}_{xx} \\ &\tilde{u}(x,0) = \tilde{f}(x) \qquad x>0 \\ &\tilde{u}(0,t) = 0 \qquad t>0 \end{aligned}\right\}$$

needed in order to have the odd extension

Apply the Fourier transform

$$\frac{\partial}{\partial t}\hat{\tilde{u}}(\xi,t) = D(-i\xi)^2\hat{\tilde{u}}(\xi,t) = -D\xi^2\hat{\tilde{u}}(\xi,t)$$

$$\hat{\tilde{u}}(\xi,t) = c(\xi)e^{-D\xi^2 t}$$

$$\hat{\tilde{u}}(\xi,0) = \hat{\tilde{f}}(\xi) = c(\xi)e^{-D\xi^2 0} = c(\xi)$$

$$\Rightarrow \hat{\tilde{u}}(\xi, t) = \hat{\tilde{f}}(\xi) e^{-\frac{\xi^2}{\left(\frac{1}{Dt}\right)}}$$

$$\Rightarrow \tilde{u}(x,t) = \mathcal{F}^{-1}\left(\hat{\tilde{f}}(\xi) e^{-\frac{\xi^2}{\left(\frac{1}{Dt}\right)}}\right) =$$

$$= \tilde{f}(x) * \mathcal{F}^{-1}\left(e^{-\frac{\xi^2}{4\left(\frac{1}{4Dt}\right)}}\right)$$

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}} \qquad \text{Put } a = \frac{1}{4Dt}$$

$$\mathcal{F}\left(\frac{1}{\sqrt{\pi 4Dt}} e^{-\frac{1}{4Dt}x^2}\right) = \frac{1}{\cancel{\sqrt{\pi 4Dt}}} \cancel{\sqrt{\frac{\pi}{\left(\frac{1}{4Dt}\right)}}}\, e^{-\frac{\xi^2}{4\left(\frac{1}{4Dt}\right)}}$$

$$\Rightarrow \mathcal{F}^{-1}\left(e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{1}{4Dt}x^2}$$

$$\hat{\tilde{u}}(\xi, t) = \hat{\tilde{f}}(\xi)\, \mathcal{F}\left(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{1}{4Dt}x^2}\right) \Rightarrow$$

$$\tilde{u}(x,t) = \tilde{f}(x) * \left(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{1}{4Dt}x^2}\right) =$$

$$= \int_{-\infty}^{\infty} \tilde{f}(y) \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{1}{4Dt}(x-y)^2}\, dy$$

and $u(x,t)$ is the restriction of $\tilde{u}(x,t)$ to the first quadrant ☺