Math 46 Solutions to homework problems Day 26

Exercise 5 page 396

Verify the following properties of the Fourier transform

(a) $(\mathcal{F}u)(\xi) \overset{?}{=} 2\pi(\mathcal{F}^{-1}u)(-\xi)$

$$(\mathcal{F}u)(\xi) = \int_{-\infty}^{\infty} u(x)e^{i\xi x}\,dx$$

$$2\pi(\mathcal{F}^{-1}u)(-\xi) = 2\pi\,\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{-i(-\xi)x}\,dx = \int_{-\infty}^{\infty} u(x)e^{i\xi x}\,dx$$

(b) $\mathcal{F}(e^{iax}u(x))(\xi) \overset{?}{=} \hat{u}(\xi+a)$

$$\int_{-\infty}^{\infty} e^{iax}u(x)e^{i\xi x}\,dx = \int_{-\infty}^{\infty} u(x)e^{i(\xi+a)x}\,dx$$

(c) $\mathcal{F}(u(x+a)) \overset{?}{=} e^{-ia\xi}\hat{u}(\xi) = e^{-ia\xi}\int_{-\infty}^{\infty} u(x)e^{ix\xi}\,dx$

$$\mathcal{F}(u(x+a)) = \int_{-\infty}^{\infty} u(x+a)e^{ix\xi}\,d\cancel{x} = \int_{-\infty}^{\infty} u(\tilde{x})e^{i(\tilde{x}-a)\xi}\,dx$$

Put $\tilde{x} = x+a$

$x = \tilde{x} - a$

Exercise 6 page 396

Find the Fourier transform of the following functions.

(a) $u(x) = H(x)\, e^{-ax}$

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{i\xi x}\, dx =$$

$$= \int_{-\infty}^{\infty} H(x) e^{-ax} e^{i\xi x}\, dx =$$

$$= \int_{0}^{\infty} 1\, e^{x(i\xi - a)}\, dx = \lim_{R\to\infty} \int_{0}^{R} e^{x(i\xi-a)}\, dx$$

$$= \lim_{R\to\infty} \frac{1}{i\xi - a} e^{x(i\xi - a)} \Big]_{x=0}^{x=R} =$$

$$= \underbrace{\lim_{R\to\infty} \frac{1}{i\xi - a} e^{-aR} \left(\cos(x\xi) + i\sin(x\xi)\right)}_{\xrightarrow{R\to\infty} 0} -$$

$$- \frac{1}{i\xi - a} e^{0(i\xi - a)} = \frac{1}{a - i\xi}$$

$$\Rightarrow \hat{u}(\xi) = \frac{1}{a - i\xi}$$

(b) $u(x) = x e^{-ax^2}$

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} x e^{-ax^2} e^{i\xi x}\, dx =$$

$$= \lim_{R\to\infty} \int_{-R}^{R} x e^{-ax^2} e^{i\xi x}\, dx =$$

$$\left(x e^{-ax^2}\right) = \left(\frac{-1}{2a} e^{-ax^2}\right)'$$

$$= \lim_{R\to\infty} \left(-\frac{1}{2a} e^{-ax^2} e^{i\xi x} \Big]_{x=-R}^{x=R} + \right.$$

goes to 0 as $x\to\infty$

is equal to 1 in absolute value

$$\left. + \frac{1}{2a} \int_{-R}^{R} i\xi\, e^{-ax^2} e^{i\xi x}\, dx \right) =$$

$$= \frac{i\xi}{2a} \int_{-\infty}^{\infty} e^{-ax^2} e^{i\xi x}\, dx =$$

$$= \frac{i\xi}{2a} \mathcal{F}\left(e^{-ax^2}\right)(\xi) = \frac{i\xi}{2a} \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}$$

☺