Corrected on May 25

Math 46 Homework Solutions

Day 24

Exercise 1 part a page 38!

(i) Use eigenfunction expansion method to solve the following problems.

$$\begin{cases} (1)\ u_t = k u_{xx} & 0<x<\ell \quad t>0 \\ (2)\ u_x(0,t) = u_x(\ell,t) = 0 & t>0 \\ (3)\ u(x,0) = f(x) & 0<x<\ell \end{cases}$$

We search for solutions

$$u(x,t) = X(x)\,T(t)$$

(1) becomes $X(x)T'(t) = k X''(x) T(t)$

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{depends only on } x} = \underbrace{\frac{T'(t)}{kT(t)}}_{\text{depends only on } t}$$

Thus both sides are a constant function say $-\lambda$

$$\left.\begin{aligned} \frac{X''(x)}{X(x)} &= -\lambda \\ \frac{T'(t)}{kT(t)} &= -\lambda \end{aligned}\right\} \quad \begin{aligned} &(2) \text{ gives} \\ &X'(0)T(t) = 0 \quad \forall t \\ &X'(\ell)T(t) = 0 \end{aligned}$$

If $T(t) \equiv 0$ then we do get the trivial solution $u(x,t) \equiv 0$ which is not going to be helpful to us.

$\Rightarrow$ $X'(0) = X'(\ell) = 0$ Thus we get

$$\left.\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) = X'(\ell) &= 0 \end{aligned}\right\} \ (*)$$

From our previous experience we know that solution exists

when $\lambda = 0 \Longleftrightarrow X_0(x) = 1$

when $\lambda = \left(\frac{\pi n}{\ell}\right)^2$ $X_n(x) = \cos\left(\frac{\pi n x}{\ell}\right)$

$n = 1, 2, 3, 4, 5, \ldots$

Now the other equation

$\frac{T'(t)}{kT(t)} = -\lambda$ becomes

$\lambda = 0$ $\frac{T'(t)}{kT(t)} = -0 \Rightarrow T_0(t) = 1$

Thus $u_0(x,t) = X_0(x)T_0(t) = 1 \cdot 1 = 1$

$\lambda = \left(\frac{\pi n}{\ell}\right)^2$ $T'(t) = -\left(\frac{\pi n}{\ell}\right)^2 kT(t)$

$\Rightarrow T_n(t) = e^{-k\left(\frac{\pi n}{\ell}\right)^2 t}$

$u_n(x,t) = \cos\left(\frac{\pi n x}{\ell}\right) e^{-k\left(\frac{\pi n}{\ell}\right)^2 t}$ $n = 1, 2, 3$

$u_0(x,t) = 1$

Consider the formal solution series $C_0 \cdot u_0 + \sum_{n=1}^{\infty} C_n u_n = u(x,t)$

$$C_0 \cdot 1 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{\ell}\right) e^{-k\left(\frac{\pi n}{\ell}\right)^2 t}$$

We want to get that

$$u(x,0) = f(x)$$

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty} c_n \cos\left(\frac{\pi n x}{\ell}\right) \underbrace{e^{-k\left(\frac{\pi n}{\ell}\right)^2 \cdot 0}}_{=1}}_{=f(x)}$$

$$\Rightarrow C_0 1 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{\ell}\right) = f(x)$$

orthonormal on $[0,\ell]$

as we know

$$C_0 = \frac{(1,f)}{(1,1)} = \frac{\int_0^\ell 1 f(x)\,dx}{\int_0^\ell 1 \cdot 1\,dx} = \frac{1}{\ell}\int_0^\ell f(x)\,dx$$

$$C_n = \frac{\left(\cos\left(\frac{\pi n x}{\ell}\right), f(x)\right)}{\left(\cos\left(\frac{\pi n x}{\ell}\right), \cos\left(\frac{\pi n x}{\ell}\right)\right)}$$

$$\int_0^{\ell} \cos\left(\frac{\pi n x}{\ell}\right)^2 dx = \int_0^{\ell} \frac{1+\cos\left(2\left(\frac{\pi n x}{\ell}\right)\right)}{2}\, dx$$

$$2\cos^2\alpha - 1 = \cos 2\alpha \Rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \frac{1}{2}x + \underbrace{\frac{\ell}{2\pi n}\sin\left(\frac{2\pi n x}{\ell}\right)}_{=0}\Bigg]_{x=0}^{x=\ell} = \frac{1}{2}\ell$$

$$\Rightarrow \quad C_n = \frac{\int_0^{\ell} f(x)\cos\left(\frac{\pi n x}{\ell}\right) dx}{\frac{1}{2}\ell}$$

$n = 1, 2, 3 \ldots$

The answer is

$$u(x,t) = C_0 \cdot 1 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{\ell}\right) e^{-k\left(\frac{\pi n}{\ell}\right)^2 t}$$

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Exercise 1 part b page 382

$$u_{tt} = c^2 u_{xx} - a^2 u \quad 0<x<\ell \quad t>0$$
$$u(0,t) = u(\ell,t) = 0 \quad t>0$$
$$u(x,0) = f(x), \; u_t(x,0) = 0 \quad 0<x<\ell$$

Solution we search for solutions $u(x,t) = X(x)T(t)$ s.t.

① $u_{tt} = c^2 u_{xx} - a^2 u$

② $u(0,t) = u(\ell,t) = 0 \quad t>0$

③ $u_t(x,0) = 0 \quad 0<x<\ell$

and then make a series out of them and search for coefficients so that $u(x,0) = f(x)$

① gives $X(x)T''(t) = c^2 X''(x)T(t) - a^2 X(x)T(t)$

$$c^2 X''(x)T(t) = X(x)\left(T''(t) + a^2 T(t)\right)$$

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{depends only on } x} = \underbrace{\frac{T''(t) + a^2 T(t)}{c^2 T(t)}}_{\text{depends only on } t}$$

Thus both sides are constants say $-\mu$

$$\left.\begin{array}{l} \dfrac{X''(x)}{X(x)} = -\mu \quad (\mathrm{I}) \\ (\mathrm{II}) \quad \dfrac{T''(t) + a^2 T(t)}{c^2 T(t)} = -\mu \end{array}\right\}$$

$$\left.\begin{array}{l} u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \\ u(\ell,t) = 0 \Rightarrow X(\ell)T(t) = 0 \end{array}\right\}$$

if $T(t) \equiv 0$ then we get the trivial solution. Thus we get the conditions $X(0) = X(\ell) = 0$

(I) + these conditions give us

$$\left.\begin{array}{l} X''(x) + \mu X(x) = 0 \\ X(0) = X(\ell) = 0 \end{array}\right\}$$

Solutions exist only when

$$\mu_n = \left(\frac{\pi n}{\ell}\right)^2 \longleftrightarrow X_n(x) = \sin\left(\frac{\pi n x}{\ell}\right)$$

(II) becomes

$$\frac{T''(t) + a^2 T(t)}{c^2 T(t)} = -\left(\frac{\pi n}{\ell}\right)^2$$

$$T_n''(t) + \left(a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2\right) T_n(t) = 0$$

Condition ③ says that

$u_t(x,0) = 0 \quad X(x)T'(0) = 0$

if $X(x) \equiv 0$ then we shall get the trivial solution Thus

$T'(0) = 0$

$$\left.\begin{aligned} &T_n''(t) + \left(a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2\right) T_n(t) = 0 \\ &T_n'(0) = 0 \end{aligned}\right\}$$

$$T_n(t) = c_1 \cos\left(\sqrt{a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2}\, t\right) + c_2 \sin\left(\sqrt{a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2}\, t\right)$$

$T_n'(0) = 0 \Rightarrow c_2 = 0$

Choose $T_n(t) = \cos\left(\sqrt{a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2}\, t\right)$

Thus we get

$$u_n(x,t) = X_n(x) T_n(t) = \sin\left(\frac{\pi n x}{\ell}\right) \cos\left(\sqrt{a^2 + c^2\left(\frac{\pi n}{\ell}\right)^2}\, t\right)$$

and we get the formal solution series

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n x}{\ell}\right) \cos\left(\sqrt{a^2 + c^2\left(\frac{\pi n}{\ell}\right)^2}\, t\right)$$

Now we look at the condition (page 8)

$u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n x}{\ell}\right) \underbrace{\cos\left(\sqrt{a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2}\, 0\right)}_{=1}$$

Thus $f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n x}{\ell}\right)$

orthogonal complete set on $(0,\ell)$

$$\text{Thus } c_n = \frac{\left(f, \sin\frac{\pi n x}{\ell}\right)}{\left(\sin\left(\frac{\pi n x}{\ell}\right), \sin\left(\frac{\pi n x}{\ell}\right)\right)} =$$

$$= \frac{\int_0^{\ell} f(x) \sin\left(\frac{\pi n x}{\ell}\right) dx}{\frac{1}{2}\ell} =$$

$$= \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{\pi n x}{\ell}\right) dx = c_n \quad (*)$$

and the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n x}{\ell}\right) \cos\left(\sqrt{a^2 + \left(\frac{\pi n}{\ell}\right)^2 c^2}\, t\right)$$

with c_n given by $(*)$