Math 46 Homework Solutions

Day 23

Exercise 1 Solve the equation

$$\Omega = [0,\pi]\times[0,\pi]$$

$$-\Delta u = 1 \qquad x \in \Omega$$
$$u = 0 \qquad x \in \partial\Omega$$

As we know from example 6.21 The eigenvalues are $\lambda_{n,k} = n^2+k^2$ corresponding to the eigenfunctions

$$u_{n,k} = \sin nx \sin ky$$

Now $$1 = \sum_{n,k=1}^{\infty} c_{nk} \sin nx \sin ky$$

for $$c_{n,k} = \frac{(1, u_{nk})}{(u_{nk}, u_{nk})} =$$

$$= \frac{\int_\Omega 1 \sin nx \sin ky \, dx\,dy}{\int_\Omega (\sin nx \sin ky)^2 \, dx\,dy} =$$

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$$\underset{\text{Fubini's Theorem}}{=} \frac{\int_0^\pi \int_0^\pi \sin nx \sin ky \, dy\, dx}{\int_0^\pi \int_0^\pi \sin^2 nx \sin^2 ky \, dy\, dx}$$

$$= \frac{\int_0^\pi -\frac{1}{k} \sin(nx) \cos(ky) \Big]_{y=0}^{y=\pi} dx}{\int_0^\pi \int_0^\pi \sin^2(nx) \left(\frac{1}{2} - \frac{\cos 2ky}{2} \right) dy\, dx}$$

$1 - 2\sin^2 \alpha = \cos 2\alpha$

$\Rightarrow \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$

$$= \frac{\int_0^\pi -\frac{1}{k} \sin(nx) (\cos(k\pi) - 1)\, dx}{\int_0^\pi \sin^2(nx) \left(\frac{y}{2} - \frac{1}{2(2k)} \sin(2ky) \right]_{y=0}^{y=\pi} dx}$$

$$= \frac{\int_0^\pi -\frac{1}{k} \sin(nx) (\cos(k\pi) - 1)\, dx}{\int_0^\pi \sin^2(nx) \frac{\pi}{2}\, dx} =$$

$$\underset{\text{for similar reasons}}{=} \frac{\frac{1}{nk} (\cos(n\pi) - 1)(\cos(k\pi - 1))}{\frac{\pi^2}{4}}$$

Now $-\Delta u = 1 = \sum_{n,k=1}^{\infty} c_{nk} u_{nk}$

$\uparrow$ we found

Assume the solution is

$u(x,y) = \sum_{n,k=1}^{\infty} \alpha_{nk} u_{nk}$

$\uparrow$ unknown

Then we get

$$-\Delta \left(\sum_{n,k=1}^{\infty} \alpha_{nk} u_{nk} \right) = \sum_{n,k=1}^{\infty} c_{nk} u_{nk}$$

$$\parallel$$

$$\sum_{n,k=1}^{\infty} \alpha_{nk} \underbrace{(-\Delta u_{nk})}_{\lambda_{nk} u_{nk}} = \sum_{n,k=1}^{\infty} \alpha_{nk} \lambda_{nk} u_{nk}$$

Thus

$$\sum_{n,k=1}^{\infty} \alpha_{n,k} \lambda_{nk} u_{nk} = \sum_{n,k=1}^{\infty} c_{n,k} u_{nk}$$

take the inner product with $u_{\tilde{n},\tilde{k}}$

to get $\alpha_{\tilde{n},\tilde{k}} \lambda_{\tilde{n},\tilde{k}} \cancel{(u_{\tilde{n},\tilde{k}}, u_{\tilde{n}\tilde{k}})} = c_{\tilde{n}\tilde{k}} \cancel{(u_{\tilde{n},\tilde{k}}, u_{\tilde{n},\tilde{k}})}$

$$\alpha_{\tilde{n}\tilde{k}} = \frac{c_{\tilde{n}\tilde{k}}}{\lambda_{\tilde{n}\tilde{k}}} = \frac{c_{\tilde{n}\tilde{k}}}{\tilde{n}^2 + \tilde{k}^2} \quad \text{hold } \forall\, \tilde{n}, \tilde{k} \in \mathbb{N}$$

Thus $\alpha_{n,k} = \dfrac{c_{n,k}}{n^2+k^2}$

Thus the solution is

$$u(x,y) = \sum_{n,k=1}^{\infty} \alpha_{n,k}\, u_{nk}(x,y) =$$

$$= \sum_{n,k=1}^{\infty} \left(\frac{4\,(\cos(n\pi)-1)(\cos(k\pi)-1)}{nk\pi^2\,(n^2+k^2)} \right) \cdot$$

$$\cdot \sin(nx)\sin(ny)$$

Exercise 2 $\Omega = [0,\pi] \times [0,\pi]$

Solve the equation

$$-\Delta u = 3 \sin 4x \sin 3y$$
$$u = 0 \quad x \in \partial\Omega$$

Solution As we know the eigenfunctions and eigenvalues are

$$\lambda_{n,k} = n^2 + k^2 \qquad n,k \in \mathbb{N}$$
$$u_{n,k}(x,y) = \sin(nx)\sin(ky)$$

Put $f(x,y) = 3 \sin 4x \sin 3y$

We can find its Fourier series

$$f(x,y) = \sum_{n,k=1}^{\infty} c_{nk}\, u_{n,k}(x,y)$$

where $c_{nk} = \dfrac{(f, u_{nk})}{(u_{nk}, u_{nk})}$

If $u = \sum_{nk=1}^{\infty} \alpha_{nk} u_{nk}(x,y)$ is a solution then we have

$$-\Delta u = f(x,y)$$

$$-\Delta\left(\sum_{n,k=1}^{\infty} \alpha_{nk} u_{nk}(x,y)\right) = \sum_{n,k=1}^{\infty} c_{nk} u_{nk}(x,y)$$

$$\sum_{nk=1}^{\infty} \alpha_{nk}\left(\underbrace{-\Delta u_{nk}(x,y)}_{=\lambda_{nk} u_{nk}(x,y)}\right)$$

$$\Rightarrow \sum_{nk=1}^{\infty} \alpha_{nk} \lambda_{nk} u_{nk}(x,y) = \sum_{n,k=1}^{\infty} c_{nk} u_{nk}(x,y)$$

By standard by now argument (explain this on the exam and when you write down such problems. Basically you take the inner product of both sides with $u_{\tilde{n}\tilde{k}}$ as in the previous problem)

We get $\quad \alpha_{nk} = \dfrac{c_{nk}}{\lambda_{nk}}$

Now one notices that

$$f(x,y) = 3\sin 4x \sin 3y = 3u_{4,3}(x,y)$$

So $c_{n,k} = \dfrac{(f,\ u_{n,k})}{(u_{nk}, u_{nk})} \neq 0$ only for $n=4$, $k=3$

$$\parallel$$

$$\frac{(3u_{43},\ u_{nk})}{(u_{nk},\ u_{nk})}$$

$$c_{4,3} = \frac{3(u_{43}, u_{43})}{(u_{4,3}, u_{43})} = 3$$

Thus c_{43} is the only nonzero Fourier coefficient in the decomposition of f.

$$\alpha_{4,3} = \frac{c_{4,3}}{\lambda_{4,3}} = \frac{3}{4^2+3^2}$$ and the other coefficients are zero

Thus $u(x,y) = \dfrac{3}{4^2+3^2} \sin(4x)\sin(3y)$

Exercise 3

Find the eigenvalues and the eigenfunctions corresponding to the homogeneous Neumann problem,

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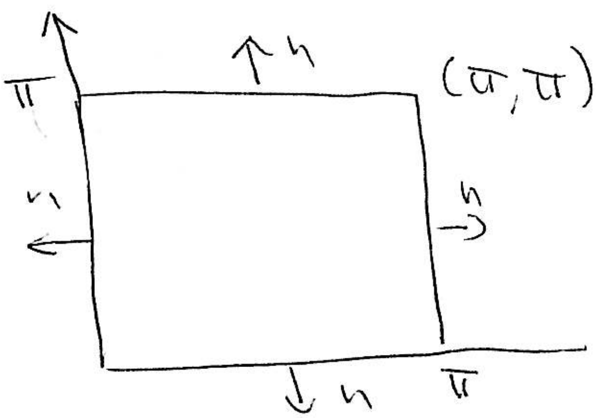
$$-\Delta u = \lambda u \quad \forall x \in \Omega = [0,\pi]\times[0,\pi]$$

$$\frac{\partial u}{\partial n} = 0 \qquad \forall x \in \partial\Omega$$



We look for

$$u(x,y) = X(x)\,Y(y)$$

so the equation becomes

$$-u_{xx} - u_{yy} = \lambda u$$

$$\frac{\partial u}{\partial x}(\pi, y) = 0$$

$$\frac{\partial u}{\partial y}(x,\pi) = 0 \qquad -\frac{\partial u}{\partial y}(x,0) = 0$$

$$-\frac{\partial u}{\partial x}(0,y) = 0$$

Note the "−" signs!!!

$$-X''(x)Y(y) - X(x)Y''(y) = \lambda X(x)Y(y) \quad (*)$$

$$\frac{du}{dx}(0,y) = \frac{du}{dx}(\pi, y) = 0$$

$$\forall y \in [0,\pi] \quad (\triangle)$$

$$\frac{du}{dy}(x,0) = \frac{du}{dy}(x,\pi) = 0$$

$$\forall x \in [0,\pi]$$

Now we work with $(*)$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y) + \lambda Y(y)}{Y(y)}$$

depends only on x — depends only on y

Thus both sides are constants

$$\left.\begin{aligned} -\frac{X''}{X} &= \mu \quad (1) \\ \frac{Y'' + \lambda Y}{Y} &= \mu \quad (2) \end{aligned}\right\}$$

(1) gives $X'' + \mu X = 0$

Now from (Δ) we get

$$\left.\begin{aligned} X'(0)\,Y(y) &= 0 \\ X'(\pi)\,Y(y) &= 0 \end{aligned}\right\}$$

if $Y(y) \equiv 0$ then u is trivial and this is not what we want

$\Rightarrow X'(0) = X'(\pi) = 0$

similarly $$\left.\begin{aligned} X(x)\,Y'(0) &= 0 \\ X(x)\,Y'(\pi) &= 0 \end{aligned}\right\}$$

give the only interesting answer

$Y'(0) = Y'(\pi) = 0$

$$\left.\begin{aligned} X'' + \mu X &= 0 \\ X'(0) = X'(\pi) &= 0 \end{aligned}\right\}$$ is an SLP

we know that it has the following solutions if $\mu = 0$

$\Rightarrow X_0 = 1$

if $\mu = n^2 \quad n > 0$

Then $X_n = \cos(nx)$

If $\mu = 0$ then

$$\left.\begin{array}{l}\dfrac{Y'' + \lambda Y}{Y} = 0 \quad \angle \mu \\ Y'(0) = Y'(\pi)\end{array}\right\}\text{becomes}$$

$Y'' + \lambda Y = 0$ that has a
eigenfunctions
solution for $\lambda = 0$ that is

$Y_0(y) = 1$ and for $\lambda = k^2 > 0$
$k \in \mathbb{N}$

$Y_k(y) = \cos(ky)$

$\lambda = 0^2 + k^2$

If $\mu = 0$ then we get

$Y'' + \lambda Y = n^2 Y \Rightarrow Y'' + (\lambda - n^2) Y = 0$

$Y'(0) = Y'(\pi) = 0$

has a solution only when

$\lambda - n^2 = k^2$ $k \in \mathbb{N}$
or $k = 0$

if $k = 0$ it is $Y_0(y) = 1$

if $k \in \mathbb{N}$ it is $Y_k(y) = \cos ky$

Thus eigen functions and
eigenvalues are as follows

$\lambda_{00} = 0^2+0^2 \iff u_{00} = 1\cdot 1$

$\lambda_{0k} = 0^2+k^2 \iff u_{0k} = 1 \cos ky$

$\lambda_{n0} = n^2+0^2 \iff u_{nk} = \cos nx \cdot 1$

$\lambda_{nk} = n^2+k^2 \iff u_{nk}(xy) = \cos nx \cos ky$

$n, k \in \mathbb{N}$