Math 46 Solutions to homework problems Day 22 page 1

Exercise 4 page 372

$q>0$ $q \in C^0(\overline{\Omega})$ $f \in C^0(\overline{\Omega})$

$\Omega \subset \mathbb{R}^n$

$g \in C^0(\partial\Omega)$ prove that the Neumann problem

$$-\Delta u + q(\vec{x})u = f(\vec{x}) \qquad x \in \Omega$$

$$\frac{du}{dn} = g(x) \qquad x \in \partial\Omega$$

can have at most one solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$

Solution: assume $u_1(\vec{x})$ and $u_2(\vec{x})$ are two solutions. Put $w(\vec{x}) = u_1(\vec{x}) - u_2(\vec{x})$

$$-\Delta w + q(\vec{x})w = -\Delta(u_1 - u_2) + q(\vec{x})(u_1 - u_2)$$

$$= \left(-\Delta u_1 + q(\vec{x})u_1\right) - \left(-\Delta u_2 + q(\vec{x})u_2\right) =$$

$$= (f(\vec{x}) - f(\vec{x})) = 0$$

$$\frac{dw}{dn} = \frac{d(u_1 - u_2)}{dn} = \frac{du_1}{dn} - \frac{du_2}{dn} = \underbrace{g(x) - g(x)}_{=0}$$

$\Rightarrow$

page 2

$$-\Delta w + q(x)\, w = 0 \qquad \forall x \in \Omega$$

$$\frac{dw}{dn} = 0 \qquad \forall x \in \partial\Omega$$

first green's identity is

$$\int_\Omega u\Delta w + \nabla u \cdot \nabla w \, d\vec{x} = \int_{\partial\Omega} u \frac{dw}{dn} \, dA$$

use it with $u = w$

$$\int_\Omega w \underbrace{\Delta w}_{q(x)w} + \nabla w \nabla w \, dx = \int_{\partial\Omega} 1 \underbrace{\frac{dw}{dn}}_{=0} \, dA$$

since $-\Delta w + q(x)\, w = 0$ on $\partial\Omega$

$\Rightarrow$

$$\int_\Omega \underbrace{q(x) w^2}_{\geq 0} + \underbrace{\nabla w \cdot \nabla w}_{\geq 0} \, d\vec{x} = 0$$

$\Rightarrow$ $q(x)\, w^2 + \nabla w \nabla w \equiv 0$

at all points

$\Rightarrow$ ∇w is the zero vector field at all points $\Rightarrow w(\vec{x})$ is a constant function

but $q(\vec{x})\, w^2(\vec{x}) = 0$. $\forall \vec{x} \in \Omega$

Consider $\vec{x}_0$ where $q(\vec{x}_0) > 0$
To ge that $w(x_0) = 0$ and hence w is the zero function

$$\Rightarrow \quad u_1(\vec{x}) = u_2(\vec{x}) \quad \forall x \in \Omega$$

Exercise 5

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Find all the solutions of the Laplace equation $\Delta u = 0$ that are functions of r only

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi \underbrace{u_\varphi}_{=0}) + \frac{1}{r^2 \sin^2\varphi}\underbrace{u_{\theta\theta}}_{=0}$$

since we look for solutions that do not depend on φ, θ

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial r}(r^2 u_r) = 0$$

$$\Rightarrow \quad r^2 u_r = A \leftarrow \text{some constant}$$

$$\Rightarrow u_r = \frac{A}{r^2} \quad \Rightarrow u = -\frac{A}{r} + B$$

$$u(x_1, x_2, x_3) = -\frac{A}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + B$$

Exercise 6 page 372

Prove that if the Dirichlet problem
$$-\Delta u = \lambda u \quad (*) \quad x \in \Omega$$
$$u = 0 \quad x \in \partial\Omega$$
has a nontrivial solution $\Rightarrow \lambda > 0$.

Proof: We have to show that

(A) λ can not be zero

(B) λ can not be negative.

(A) $\lambda = 0$ This is the Laplace equation
$$\Delta u = 0$$
$$u = 0 \quad x \in \partial\Omega$$
It has ~~a unique~~ at most one solution by Theorem 6.18 and $u(\vec{x}) = 0$ is clearly a solution. So if $\lambda = 0$ then there are no nontrivial solution

(B) $\lambda < 0$

Green's first identity
$$\int_\Omega u\,\Delta w + \nabla u \cdot \nabla w \, d\vec{x} = \int_{\partial\Omega} u \frac{dw}{dn}\, dA$$
use it with $w = u$
$$\int_\Omega u \underbrace{\Delta u}_{-\lambda u \text{ by the } (*)} + \nabla u \cdot \nabla u \, d\vec{x} = \underbrace{\int_{\partial\Omega} u \frac{du}{dn}\, dA}_{=0}$$
is zero at points of $\partial\Omega$

$$\Rightarrow \int_{\Omega} \frac{-\lambda}{\nu_0} u^2 + \nabla u \cdot \nabla u \, dx = 0$$

$$\Rightarrow -\lambda u^2 + \frac{\nabla u \cdot \nabla u}{\nu_0} \equiv 0 \Rightarrow$$

$$\Rightarrow \frac{-\lambda u^2}{\neq 0} \equiv 0 \quad \forall x \Rightarrow u \equiv 0$$

is the zero function