Math 46 Solutions of homework problems Day 21

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Exercise 10 page 366

$$u_t - Du_{xx} = f(x) \qquad 0<x<\ell \quad t>0$$

$$\left.\begin{aligned} u_x(0,t) &= A \\ u_x(\ell,t) &= B \end{aligned}\right\} t>0$$

$$u(x,0) = u_0(x) \qquad 0<x<\ell$$

Show that if is a solution is independent of time i.e. $u(x,t)=u(x)$

$$\Rightarrow DA - DB = \int_0^\ell f(x)\,dx$$

Solution $\quad u = u(x) \Rightarrow u_t = 0$

$$\Rightarrow \quad 0 - Du_{xx} = f(x)$$

$$-\int_0^\ell Du_{xx}\,dx = \int_0^\ell f(x)\,dx$$

// by parts

$$-Du_x(x,t)\Big]_{x=0}^{x=\ell} + \int_0^\ell u_x \underbrace{\frac{d}{dx}D}_{=0}\,dx =$$

$$= -Du_x(\ell,t) + Du_x(0,t) = DA - DB$$

Exercise 13 page 367

Show that the nonlinear boundary value problem

$$u_t = u_{xx} - u^3 \quad 0<x<\ell \quad t>0$$
$$u(0,t)=0 \; u(\ell,t)=0 \quad t>0$$
$$u(x,0)=0 \quad 0<x<\ell$$ has

only the trivial solution

Proof: Take $u(x,t)=0 \Rightarrow$

$$0_t = 0_{xx} - 0^3 \text{ true}$$

and the boundary conditions are also true. Let us use the energy method to show that a solution is unique. Let $u(t)$ be a solution

Put $E(t) = \int_0^\ell u^2(x,t)\,dx$

$$E(t) \geq 0 \quad \forall t$$
$$E(0) = \int_0^\ell u^2(x,0)\,dx = 0$$
$$\frac{d}{dt}E(t) = \int_0^\ell u(x,t)\,u_t(x,t)\,dx$$
$$= \int_0^\ell u(x,t)(u_{xx} - u^3)\,dx =$$

$$= \int_0^\ell u\, u_{xx}\, dx - \int_0^\ell u^4\, dx =$$

$$= u(x,t)\, u_x(x,t)\Big]_{x=0}^{x=\ell} - \int_0^\ell (u_x)^2\, dx -$$

$$- \int_0^\ell u^4\, dx = \underbrace{u(\ell,t)}_{=0}\, u_x(\ell,t) - \underbrace{u(0,t)}_{=0}\, u_x(0,t)$$

$$- \underbrace{\int_0^\ell (u_x)^2\, dx}_{\geq 0} - \underbrace{\int_0^\ell u^4\, dx}_{\geq 0} \Rightarrow$$

$$\Rightarrow \frac{dE}{dt}(t) \leq 0 \quad \forall t \quad \Rightarrow$$

$$\left.\begin{array}{l} E(0) = 0 \\ E(t) \geq 0 \quad \forall t \\ E(t) \text{ is nonincreasing} \end{array}\right) \Rightarrow E(t) = 0 \quad \forall t$$

$$\Rightarrow \quad 0 = E(t) = \int_0^\ell u^2(x,t)\, dx$$

$$\Rightarrow \forall t \quad u(x,t) = 0 \quad \forall x \Rightarrow$$

$$\Rightarrow \quad u(x,t) = 0 \quad \forall x, \forall t$$