Solutions of homework problems
Day 20

Exercise 2 page 365

use energy method to prove that a solution to the initial boundary value problem

$$\left.\begin{array}{ll} u_t - k u_{xx} = 0 & 0<x<\ell \\ & 0<t<T \\ u(x,0) = f(x) & \\ u_x(0,t) = 0 \quad u(\ell,t) = h(t) & \end{array}\right\} (*)$$

must be unique.

Solution Assume that we have two solutions $u_1(x,t)$, $u_2(x,t)$ of $(*)$ $\Rightarrow$ put $w(x,t) = u_1(x,t) - u_2(x,t)$

$$w_t - k w_{xx} = (u_1-u_2)_t - k(u_1-u_2)_{xx} =$$
$$= \underbrace{(u_{1t} - k u_{1xx})}_{=0} - \underbrace{(u_{2t} - k u_{2xx})}_{=0} = 0$$

$$w(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x)$$

$$w_x(0,t) = u_{1x}(0,t) - u_{2x}(0,t) = 0 - 0 = 0$$

$$w(\ell,t) = u_1(\ell,t) - u_2(\ell,t) = h(t) - h(t) = 0$$

Thus

$$\left.\begin{array}{ll} w_t - k w_{xx} = 0 & \\ w(x,0) = 0 & w(\ell,t) = 0 \\ w_x(0,t) = 0 & \end{array}\right\} (**)$$

Put $E(t) = \int_0^\ell w^2(x,t)\,dx$

$$E(0) = \int_0^\ell w^2(x,0)\,dx = \int_0^\ell 0\,dx = 0$$

$E(t) \geq 0 \ \forall t$ since it is defined as the integral of a nonnegative function $w^2(x,t)$.

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^\ell w^2(x,t)\,dx = \int_0^\ell \frac{d}{dt} w^2(x,t)\,dx =$$

$$= \int_0^\ell 2w(x,t)w_t(x,t)\,dx = \int_0^\ell 2w(x,t)k\,w_{xx}(x,t)\,dx$$

$$= 2w(x,t)kw_x(x,t)\Big]_{x=0}^{x=\ell} - 2k\int_0^\ell w_x^2(x,t)\,dx$$

↑
by parts

$$2\underbrace{w(\ell,t)}_{=0}kw_x(\ell,t) - 2w(0,t)k\underbrace{w_x(0,t)}_{=0} -$$

$$-2k\int_0^\ell w_x^2(x,t)\,dx \leq 0$$

Thus $E(0) = 0$

$E(t) \geq 0 \ \forall t$

$\frac{dE}{dt}(t) \leq 0 \ \forall t \Rightarrow E(t)$ is nonincreasing

$\Rightarrow$ $E(t)$ is the zero function

$\Rightarrow \forall t \ E(t) = 0 = \int_0^\ell \underbrace{w^2(x,t)}_{\geq 0}dx$ continuous

$\Rightarrow \forall t$ $w(x,t)$ is the zero function

$\Rightarrow$

$\Rightarrow \; w(x,t) = 0 \quad \forall x \; \forall t \quad \Rightarrow$

$u_1(x,t) = u_2(x,t) \quad \forall x, \forall t \quad \Rightarrow$ the solution is unique

Exercise 3 page 365

Use the energy method to prove the uniqueness of a solution for the problem

$$\Delta \quad \left.\begin{aligned} & u_t = \Delta u \quad \vec{x} \in \Omega \subset \mathbb{R}^m, \; t > 0 \\ & u(\vec{x},0) = f(\vec{x}) \quad \vec{x} \in \Omega \\ & u(\vec{x},t) = g(\vec{x}) \quad \vec{x} \in \partial\Omega, \; t > 0 \end{aligned}\right\}$$

Solution Assume there are two solutions $u_1(\vec{x},t)$ $u_2(\vec{x},t)$ for Δ

Put $w(\vec{x},t) = \vec{u}_1(\vec{x},t) - u_2(\vec{x},t)$

$$w_t = u_{1t} - u_{2t} = \Delta u_1 - \Delta u_2 = \Delta(u_1 - u_2) = \Delta w$$

$$w(\vec{x},0) = u_1(\vec{x},0) - u_2(\vec{x},0) = f(\vec{x}) - f(\vec{x}) = 0$$

for $\vec{x} \in \partial\Omega$ $\quad w(\vec{x},t) = u_1(\vec{x},t) - u_2(\vec{x},t) = g(x) - g(x) = 0$

$$\Rightarrow \left.\begin{aligned} & w_t = \Delta w \\ & w(\vec{x},0) = 0 \quad \forall \vec{x} \in \Omega \\ & w(\vec{x},t) = 0 \quad \forall x \in \partial\Omega, \; t \geq 0 \end{aligned}\right\} \Delta\Delta$$

Put $E(t) = \int_\Omega \underbrace{w^2(\vec{x},t)}_{\geq 0} d\vec{x} \Rightarrow E(t) \geq 0 \ \forall t$

$$E(0) = \int_\Omega \underbrace{w^2(\vec{x},0)}_{=0} dx = 0$$

$$\frac{dE}{dt}(t) = \frac{d}{dt}\int_\Omega w^2(\vec{x},t)\,d\vec{x} = \int_\Omega \frac{d}{dt}\left(w^2(\vec{x},t)\right) d\vec{x} =$$

$$= \int_\Omega 2w(\vec{x},t)\underbrace{w_t(\vec{x},t)}_{=\Delta w(\vec{x},t)}\,d\vec{x} = \int_\Omega 2w(\vec{x},t)\,\Delta w(\vec{x},t)\,d\vec{x}$$

$$= \int_\Omega 2w(\vec{x},t)\sum_{i=1}^m \frac{\partial^2 w(\vec{x},t)}{\partial x_i^2}\,d\vec{x} =$$

$$= \sum_{i=1}^m \int_\Omega 2w(\vec{x},t)\frac{\partial^2 w}{\partial x_i^2}(\vec{x},t)\,d\vec{x} =$$

$$= \sum_{i=1}^m \int_{\partial\Omega} \underbrace{2w(\vec{x},t)}_{=0}\frac{\partial w}{\partial x_i}(\vec{x},t)\,n_i\,dA -$$

integration by parts on page 352

$\forall \vec{x} \in \partial\Omega$

i-th component of the unit length vector $\vec{n}$ orthogonal to $\partial\Omega$ and pointing outside of Ω

$$- \sum_{i=1}^m \int_\Omega 2\frac{\partial w}{\partial x_i}(\vec{x},t)\frac{\partial w}{\partial x_i}(\vec{x},t)\,d\vec{x}$$

$$= -\sum_{i=1}^m \int_\Omega 2\left(\frac{\partial w}{\partial x_i}(\vec{x},t)\right)^2 d\vec{x} \leq 0$$

$\Rightarrow$ $E(0) = 0$ $\quad E(t) \geq 0$

$$\frac{dE}{dt}(t) \leq 0 \ \forall t \Rightarrow E(t) \equiv 0$$

$$\Rightarrow \quad \forall t \quad 0 = E(t) = \int_{\Omega} w^2(\vec{x},t)\,dt$$

$$\Rightarrow \quad \forall t \Rightarrow \quad w(\vec{x},t) = 0 \quad \forall \vec{x} \in \Omega \Rightarrow$$

$$\forall \vec{x}, \forall t \quad w(\vec{x},t) = 0$$

$$\Rightarrow \forall \vec{x}, \forall t \quad u_1(\vec{x},t) = u_2(\vec{x},t)$$

and the solution is clearly unique.