Sketch of homework solutions

Day 1

Problem 1 page 7

P is measured in time units T
ℓ is measured in length units L
g is measured in $\frac{L}{T^2}$
m is measured in mass units M

Since one can not form a dimensionless quantity out of P, m, ℓ it is not possible that P depends only on ℓ and m.

There is one dimensionless quantity one can form out of P, ℓ, g, m
it is $\frac{P^2}{\ell} g$ measured in $\frac{T^2}{L} \frac{L}{T^2}$

so $\Pi = \frac{P^2}{\ell} g$ $[\Pi] = 1$. All other such variables are expressible through Π
Thus by the Pi-Theorem the law should be $f(\Pi) = 0$ (*)
Assuming that roots of (*) form a discrete finite set we have

$\Pi = C$ where C is one of the roots of *

$$\frac{P^2}{\ell} g = C \Rightarrow P = \underbrace{\sqrt{C}}_{\text{new constant } \tilde{C}} \sqrt{\frac{\ell}{g}} = \tilde{C} \sqrt{\frac{\ell}{g}}$$

Exercise 4 page 8

$g(t, r, \rho, e, P) = 0$

	measured in	
t		T
r	— " —	L
ρ	— " —	$\frac{M}{L^3}$
E	— " —	$\frac{M L^2}{T^2}$
P	— " —	$\underbrace{\frac{ML}{T^2}\frac{1}{L^2}}_{\text{force/area}} = \frac{M}{T^2 L}$

π_1 as suggested is $\frac{r^5 \rho}{t^2 E}$ as on page 6

π_2 should not involve r

$\pi_2 = t^x r^y \rho^z E^u p^v \qquad [\pi_2] = 1 \qquad y = 0$

$$[T^x L^0 (M^z L^{-3z})(M^u L^{2u} T^{-2u})(M^v T^{-2v} L^{-v})] = 1$$

$$\left.\begin{array}{l} T^{x-2u-2v} = 1 \\ L^{-3z+2u-v} = 1 \\ M^{z+u+v} = 1 \end{array}\right\} \qquad \begin{array}{lr} x - 2u - 2v = 0 & (a) \\ -3z + 2u - v = 0 & (b) \\ z + u + v = 0 & (c) \end{array}$$

$x = 6$
$u = -2$
$z = -3$
$v = 5$

is a solution

$$\pi_2 = \frac{t^6 p^5}{E^2 \rho^3}$$

$$\frac{r^5 \rho}{t^2 E} = g\left(\frac{t^6 p^5}{E^2 \rho^3}\right)$$

$$[\Pi_2] = \left[\frac{T^6 \left(\frac{M}{T^2 L}\right)^5}{\left(\frac{ML^2}{T^2}\right)^2 \left(\frac{M}{L^3}\right)^3}\right] = 1$$

$$r = \left(\frac{E}{\rho}\right)^{1/5} t^{2/5} \left(g\left(\frac{t^6 p^5}{E^2 \rho^3}\right)\right)^{1/5}$$

does depend on t

So one can <u>not</u> conclude that r varies as the two-fifth power of t. ☺

Problem 3 page 17

First rewrite as instructed

$$\underbrace{\frac{2}{9} r^2 \rho g \mu^{-1}\left(1-\frac{\rho_e}{\rho}\right)}_{f(r,\rho,g,\mu,\rho_e,v)} - v = 0$$

Let us look at units of measurement

r — length L

v — $\frac{L}{T^2}$ $\frac{\text{length}}{\text{time}^2}$

ρ — $\frac{M}{L^3}$ ← mass

ρ_e — $\frac{M}{L^3}$

g — $\frac{L}{T^2}$

μ — $\frac{M}{LT}$

Now assume that $\bar{L} = \lambda_1 L$

$\bar{T} = \lambda_2 T$

$\bar{M} = \lambda_3 M$

we get

$$f(\bar{r}, \bar{\rho}, \bar{g}, \bar{\mu}, \bar{\rho_e}, \bar{v}) = \frac{2}{9}\bar{r}^2 \bar{\rho}\, \bar{g}\, \bar{\mu}^{-1}\left(1-\frac{\bar{\rho_e}}{\bar{\rho}}\right) - \bar{v}$$

$$= \frac{2}{9}\left(\lambda_1^2 r^2\right)\left(\frac{\lambda_3}{\lambda_1^3}\rho\right)\left(\frac{\lambda_1}{\lambda_2^2} g\right)\left(\frac{\lambda_3}{\lambda_1\lambda_2}\right)^{-1}\mu^{-1}\left(1-\frac{\frac{\cancel{\lambda_3}}{\cancel{\lambda_1^3}}\rho_e}{\frac{\cancel{\lambda_3}}{\cancel{\lambda_1^3}}\rho}\right) -$$

$$- \frac{\lambda_1}{\lambda_2^2} v = \frac{\lambda_1}{\lambda_2}\left(\frac{2}{9} r^2 \rho g \mu^{-1}\left(1-\frac{\rho_e}{\rho}\right) - v\right) \Rightarrow$$

$\Rightarrow \quad f(\bar{r}, \bar{\rho}, \bar{g}, \bar{\mu}, \bar{\rho}_e, \bar{v}) = \frac{\lambda_1}{\lambda_2} f(r, \rho, g, \mu, \rho_e, v)$

$\Rightarrow f(\bar{r}, \bar{\rho}, \bar{g}, \bar{\mu}, \bar{\rho}_e, \bar{v}) = 0 \Leftrightarrow$

$f(r, \rho, g, \mu, \rho_e, v) = 0$ and the law is unit free.

Problem 6 page 17

Assume that there is a unit free law involving

length l — measured in L

density ρ measured in $\frac{M}{L^3}$ ← mass

resource assimilation rate a measured in $\frac{M}{L^2 T}$

resource use rate b measured in $\frac{M}{L^3 T}$

The dimension matrix is

$$\begin{array}{c} \\ L \\ M \\ T \end{array}\begin{array}{c} \begin{array}{cccc} l & \rho & a & b \end{array} \\ \begin{pmatrix} 1 & -3 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

has the same rank as $\begin{pmatrix} 1 & -3 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

i.e. rank 2. So the law is equivalent to the one involving $(4-2) = 2$ dimensionless quantities