Math 46 Homework Solutions

Day 19

__Exercise__ 2.e page 345

Find the general solution of

$uu_t = x - t \qquad u = u(x,t)$

__Clearly__ The left handside is

just $\frac{d}{dt}\left(\frac{u^2}{2}\right)$

So we have $\frac{d}{dt}\left(\frac{u^2}{2}\right) = x - t$

$$\Rightarrow \frac{u^2}{2} = \int_0^t (x-\tau)\,d\tau + A(x) =$$

$$= -\frac{1}{2}(x-\tau)^2 \Big]_{\tau=0}^{\tau=t} + A(x) =$$

$$= -\frac{1}{2}(x-t)^2 + \underbrace{\frac{1}{2}x^2 + A(x)}_{= B(x)}$$

Thus the general solution is coming from

$$\frac{u^2}{2} = -\frac{1}{2}(x-t)^2 + B(x)$$

$$u(x,t) = \pm\sqrt{\underbrace{\tilde{B}(x)}_{= 2B(x)} - (x-t)^2}$$

Exercise 5 page 346

Introduce polar coordinates $x = r\cos\theta$ $y = r\sin\theta$ to show that the general solution of the equation $yu_x - xu_y = 0$ is $u = \varphi(x^2+y^2)$

$\uparrow$ some function

$u(x,y) = u(r\cos\theta, r\sin\theta)$

$$\Rightarrow \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\underbrace{(-r\sin\theta)}_{-y} + \frac{\partial u}{\partial y}\underbrace{r\cos\theta}_{x}$$

$\uparrow$ by the chain rule

$$= \frac{\partial u}{\partial x}(-y) + \frac{\partial u}{\partial y}x$$

Thus our equation $yu_x - xu_y = 0$ is just the statement that

$$-\frac{\partial u}{\partial \theta} = 0 \quad \text{i.e.}$$

$u(x,y)$ depends only on $r = \sqrt{x^2+y^2}$ or which is the same only on $r^2 = x^2+y^2$

Thus $u(x,y) = \varphi(x^2+y^2)$ for some function φ

Exercise 7 page 346

Show that the equation

$u_t - (K(x)u_x)_x = f(t,x)$ is linear

when $K(x)$ and $f(t,x)$ are some functions.

Solution Our equation says

$Lu = f$ for $Lu = u_t - (Ku_x)_x$

We need to show that

(1) $L(u+v) \stackrel{?}{=} Lu + Lv$

$$(u+v)_t - (k(u+v)_x)_x \qquad u_t - (ku_x)_x + v_t - (kv_x)_x$$

$$= u_t + v_t - k_x(u+v)_x - k(u+v)_{xx} \qquad = u_t - k_x u_x - ku_{xx} + v_t - k_x v_x - kv_{xx}$$

$$= u_t + v_t - k_x u_x - k_x v_x - ku_{xx} - kv_{xx}$$

(2) $L(cu) \stackrel{?}{=} cLu \quad \forall c \in \mathbb{R}$

$$(cu)_t - (k(cu)_x)_x \qquad c(u_t - (ku_x)_x)$$

$$= cu_t - (kc(u_x))_x = cu_t - c(ku_x)_x$$

Thus L is linear and hence this is a linear PDE