Math 46 Solutions for homework problems Day 18

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Exercise 4 page 267

$\varphi \in C_0^\infty(a,b)$ is a test function

(a) Is it true that if $\psi_n = \frac{1}{n}\varphi(x)$ then

$\psi_n \to 0$ in $C_0^\infty(\ ,\ )$ $\qquad \psi_n^{(m)} = \frac{1}{n}\varphi^{(m)}(x)$

Now since $\varphi \in C_0^\infty(\mathbb{R})$ $\forall m$ $\exists K_m \in \mathbb{R}$ s.t. $|\varphi^{(m)}(x) - 0| < K_m$ $\forall x \in \mathbb{R}$

since $\varphi^{(m)}$ is continuous and because it has bounded support $|\varphi^{(m)}(x)|$ attains the maximal value on $\mathbb{R}$ and we can take K_m to be twice the value

Thus $|\psi_n^{(m)}(x) - 0| < \underbrace{\frac{1}{n}K_m}_{\text{goes to zero uniformly}}$ $\forall x \in \mathbb{R}$

$\Rightarrow |\psi_n^{(m)} - 0|$ also go to zero uniformly $\Rightarrow \psi_n \xrightarrow[n\to\infty]{} 0$ in $C_0^\infty(\mathbb{R})$ indeed

(b) $\varphi_n(x) = \frac{1}{n}\varphi\left(\frac{x}{n}\right)$

if the support of φ is contamed in $[-R, R]$ then the support of φ_n is contamed in $[-nR, nR]$ thus there is no bounded segment that contains the support of all φ_n simaltaneously

$\Rightarrow \varphi_n \not\to 0$

(c) $\varphi_n(x) = \frac{1}{n}\varphi(nx)$

Let us look at

$$\varphi_n'(0) = \frac{1}{n} n \varphi'(n0) = \varphi'(0)$$

Thus there is no reason why $\varphi_n'(0) \xrightarrow[n\to\infty]{} 0$ and thus

it is not true that $\varphi_n'(x) \to 0$ uniformly on x

$\Rightarrow$ it is not true that

$$\varphi_n \longrightarrow 0 \text{ in } C_0^\infty(\mathbb{R})$$

Exercise 6 page 268

Show that $u(x,\xi)=\frac{1}{2}|x-\xi|$ is a fundamental solution for $L=\frac{d^2}{dx^2}$ $\quad Lu=\frac{d^2u}{dx^2}$ in the distributional sense.

Thus we have to check that in the distributional sense $Lu=\delta(x-\xi)$

i.e

$$\left(\left(\frac{1}{2}|x-\xi|\right)'', \varphi(x)\right) \overset{?}{=} (\delta(x-\xi), \varphi(x)) \quad \forall \varphi \in C_0^\infty(\mathbb{R})$$

$$(\delta(x-\xi), \varphi(x)) = \varphi(\xi)$$

$$\left(\left(\frac{1}{2}|x-\xi|\right)'', \varphi(x)\right) = \int_{-\infty}^{\infty} \frac{1}{2}|x-\xi|\,\varphi''(x)\,dx$$

by definition of distributional derivative

$$= \int_{-\infty}^{\xi} \frac{1}{2}(\xi-x)\varphi''(x)\,dx + \int_{\xi}^{\infty}\frac{1}{2}(x-\xi)\varphi''(x)\,dx$$

$$= \underbrace{\lim_{J\to-\infty} \frac{1}{2}(\xi-x)\varphi'(x)\Big]_{x=J}^{x=\xi}}_{\text{zero since } \varphi \text{ is a test function}} -$$

$x=\xi$ ← vanishes

$$- \int_{-\infty}^{\xi}\frac{1}{2}(-1)\varphi'(x)\,dx +$$

$x=\xi$ ← vanishes

$$+ \underbrace{\lim_{J\to\infty}\frac{1}{2}(x-\xi)\varphi'(x)\Big]_{\xi}^{J}}_{\text{zero since } \varphi\in C_0^\infty(\mathbb{R})} - \int_{\xi}^{\infty}\frac{1}{2}\varphi'(x)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\xi} 1\, \varphi'(x)\,dx - \frac{1}{2}\int_{\xi}^{\infty} 1\,\varphi'(x)\,dx =$$

$$= \underbrace{\lim_{\eta\to-\infty} \frac{1}{2}\, 1\, \varphi(x)\Big]_{x=\eta}^{x=\xi}}_{\substack{= \leftarrow \text{ since } \varphi\in C_0^\infty(\mathbb{R}) \\ \frac{1}{2}\varphi(\xi)}} - \underbrace{\frac{1}{2}\int_{-\infty}^{\xi} 0\,\varphi(x)\,dx}_{=0}$$

$$- \underbrace{\frac{1}{2}\lim_{\eta\to\infty} 1\,\varphi(x)\Big]_{x=\xi}^{x=\eta}}_{\substack{-\frac{1}{2}\varphi(\xi) \\ \text{since } \varphi\in C_0^\infty(\mathbb{R})}} + \underbrace{\frac{1}{2}\int_{\xi}^{\infty} 0\,\varphi(x)\,dx}_{=0}$$

$$= \frac{1}{2}\varphi(\xi) - \left(-\frac{1}{2}\varphi(\xi)\right) = \varphi(\xi)$$

Exercise 3 page 3.45

Find a formula for the solution to the equation

$$u_{xt} = f(x,t) \qquad x,t > 0$$

that satisfies $\quad u(x,0) = g(x) \quad x > 0$
$\qquad\qquad\qquad\quad u(0,t) = h(t) \quad t > 0$

where f, g, h are given nice functions satisfying $\quad g(0) = h(0)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad g'(0) = h'(0)$

so that the conditions match

$$u(x,t) = \int_0^t \int_0^x f(y,\tau)\,dy\,d\tau + A(x) + B(t)$$

$$\underset{g(x)}{\overset{\|}{u(x,0)}} = \underbrace{\int_0^0 \int_0^x f(y,\tau)\,dy\,d\tau}_{=0} + A(x) + B(0)$$

$$\underset{h(t)}{\overset{\|}{u(0,t)}} = \underbrace{\int_0^t \int_0^0 f(y,\tau)\,dy\,d\tau}_{=0} + A(0) + B(t)$$

$\Rightarrow$

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$$\left.\begin{aligned} g(x) &= A(x) + B(0) \\ h(t) &= A(0) + B(t) \end{aligned}\right\} \ (\Delta)$$

$$g(0) = A(0) + B(0)$$
$$\|$$
$$h(0) = A(0) + B(0)$$

would be satisfied automatically if (Δ) holds

$$g'(0) = A'(0)$$
$$\|$$
$$h'(0) = B'(0)$$

So for example one can take

$$A(x) = g(x) - g(0)$$
$$B(t) = h(t).$$

Then $g(x) = \underbrace{(g(x) - g(0))}_{A(x)} + \underbrace{h(0)}_{B(0) \,=\, g(0)}$

$$h(t) = \underbrace{(g(0) - g(0))}_{A(0)} + h(t)$$

Thus the solution is

$$u(x,t) = \int_0^t \int_0^x f(y,\tau)\,dy\,d\tau + (g(x) - g(0)) + h(t)$$

Exercise 2 a page 345

$u = u(x,y)$ find the general solution of

$u_{xx} + u = 6y$

A particular solution to the equation is $u_p(x,y) = 6y$

Let us find the general solution to the homogeneous equation

$u_{xx} + u = 0$ Fix $t = \tilde{t}$ then we get

$$\frac{\partial^2}{\partial \cancel{t}^2} u(x,\tilde{y}) + \underbrace{u(x,\tilde{y})}_{g_{\tilde{y}}(x)} = 0$$

just an index

Thus we get $g''_{\tilde{y}}(x) + g_{\tilde{y}}(x) = 0$

As we know from ODE

$$g_{\tilde{y}}(x) = A_{\tilde{y}} \cos x + B_{\tilde{y}} \sin x$$

Thus we can take any functions

$$u_h(x,y) = A(y) \cos x + B(y) \sin t$$

to be the general solution of the homogeneous problem

The total solution $u(x,y) = A(y)\cos(x) + B(y)\sin t + 6y$ ☺

Exercise 2 part b

$u(x,t)$

$t\,u_{xx} - 4u_x = 0 \qquad u = u(x,t)$ Put $v = u_x$

$t\,v_x - 4v = 0 \qquad$ fix $t = \tilde{t}$

$\tilde{t}\,\dfrac{\partial v(x,\tilde{t})}{\partial x} - 4v(x,\tilde{t}) = 0$ put $g_{\tilde{t}}(x) = v(x,\tilde{t})$

$\Rightarrow \tilde{t}\,g_{\tilde{t}}'(x) - 4g_{\tilde{t}}(x) \Rightarrow$

$\dfrac{\tilde{t}}{4}\,g_{\tilde{t}}'(x) - g_{\tilde{t}}(x) = 0$

Thus from ODE we know that

$g_{\tilde{t}}(x) = A\,e^{\frac{4}{\tilde{t}}x}$

constant that depends on $\tilde{t}$

Thus $v(x,t) = A(t)\,e^{\frac{4}{t}x}$

$v = u_x$ and a similar trick tells us that

$u(x,t) = \dfrac{t}{4}A(t)\,e^{\frac{4}{t}x} + B(t)$

for arbitrary nice functions $A(t)$ $B(t)$:)