Math 46 Homework Solutions

Day 16

Exercise 1 page 267

(1) Does $(u,\varphi)=\varphi(0)^2$ define a distribution in $D'(\mathbb{R})$

Solution $(u, \overset{\mathbb{R}}{\alpha\varphi}) = ((\alpha\varphi)(0))^2$

for example if $\alpha=3$ and φ from Figure 4.3

$$\alpha(u,\varphi) = \alpha(\varphi(0))^2 \neq \alpha^2\varphi^2(0)$$

Thus this map is not linear and hence it is <u>not</u> a distribution in $D'(\mathbb{R})$

Exercise 5 page 268

Prove the following statements

(a) $x\delta'(x) = -\delta(x)$

Thus we have to show that

$\forall \varphi \in C_0^\infty$ $(x\delta'(x),\varphi) \overset{?}{=} (-\delta(x),\varphi(x))$

$$\overset{def}{=} (\delta'(x), x\varphi(x)) = -(\delta(x),(x\varphi(x))') = -(\delta(x), 1\varphi(x)+x\varphi'(x)) = -\varphi(0) - 0\varphi'(0) = -\varphi(0) = (-\delta(x),\varphi(x))$$

(b) $\alpha(x)\,\delta'(x) = -\alpha'(0)\,\delta(x) + \alpha(0)\,\delta'(x)$
for $\alpha \in C^\infty(\mathbb{R})$

Thus we have to show that
$\forall\, \varphi \in C_0^\infty$

$$(\alpha(x)\,\delta'(x), \varphi(x)) \overset{?}{=} (-\alpha'(0)\,\delta(x) + \alpha(0)\,\delta'(x), \varphi(x))$$

$$\begin{aligned}
(\alpha(x)\,\delta'(x), \varphi(x)) &= (\delta'(x), \alpha(x)\varphi(x)) \\
&= -(\delta(x), (\alpha(x)\varphi(x))') \\
&= -(\delta(x), \alpha'(x)\varphi(x) + \alpha(x)\varphi'(x)) \\
&= -\alpha'(0)\varphi(0) - \alpha(0)\varphi'(0)
\end{aligned}$$

$$\begin{aligned}
(-\alpha'(0)\,\delta(x) + \alpha(0)\,\delta'(x), \varphi(x)) &= (\delta(x), -\alpha'(0)\varphi(x)) + (\delta'(x), \alpha(0)\varphi(x)) \\
&= -\alpha'(0)\varphi(0) + (\delta(x), -(\alpha(0)\varphi(x))') \\
&= -\alpha'(0)\varphi(0) - \alpha(0)\varphi'(0)
\end{aligned}$$

are equal indeed.

Exercise 8 page 268

Compute the distributional derivative of $H(x)\cos(x)$ where H is a Heaviside function. Does the derivative exist in a weak sense

$$\left(\left(H(x)\cos(x)\right)', \varphi(x)\right) = \left(H(x)\cos(x), -\varphi'(x)\right)$$

$$= -\int_{-a}^{a} H(x)\cos(x)\varphi'(x)\,dx = -\int_{0}^{a} 1\cos(x)\varphi'(x)\,dx$$

$$= -\cos(x)\varphi(\cancel{x})\Big]_{x=0}^{x=a} + \int_{0}^{a}(-\sin x)\varphi(x)\,dx = \varphi(0) +$$

is zero since $\varphi \in C_0^\infty(-a,a)$

$$+ \int_{-a}^{a} -H(x)\sin(x)\varphi(x)\,dx$$

Thus $\left(H(x)\cos(x)\right)' = \delta(x) - H(x)\sin(x)$

is a singular distribution — is locally integrable

Nonsingular distributions form a vector space. Since $\delta(x)$ is singular and $H(x)\sin(x)$ is nonsingular we get that $\left(H(x)\cos(x)\right)'$ does <u>not</u> exist in the weak sense.

Exercise 9 page 268

Show that the Sturm-Liouville operator $Au \equiv -(pu')' + qu$ is formally self adjoint

We have to check that

$$(Au, \varphi) = (u, A\varphi) \quad \forall \varphi \in C_0^\infty(a,b)$$

$$(Au,\varphi) = (-(pu')' + qu, \varphi) = (-(pu')', \varphi) +$$
$$+ (qu, \varphi) = (pu', -(-\varphi')) + (u, q\varphi) =$$
$$= (u', p\varphi') + (u, q\varphi) = (u, -(p\varphi')') +$$
$$+ (u, q\varphi) = (u, -(p\varphi')' + q\varphi) = (u, A\varphi)$$

Exercise 11

In $\mathcal{D}'(\mathbb{R})$ compute

$$\left(\frac{d}{dx} - \lambda\right)(H(x)e^{\lambda x})$$

$$\left(\left(\frac{d}{dx} - \lambda\right) H(x)e^{\lambda x}, \varphi(x)\right) =$$
$$= \left(\frac{d}{dx}(H(x)e^{\lambda x}), \varphi(x)\right) -$$
$$- (\lambda H(x)e^{\lambda x}, \varphi(x))$$

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$= (H(x)e^{\lambda x}, -\varphi'(x)) -$

$- (\lambda H(x) e^{\lambda x}, \varphi(x)) =$

$= (H(x), -e^{\lambda x}\varphi'(x)) - (H(x), \lambda e^{\lambda x}\varphi(x))$

since $e^{\lambda x} \in C^{\infty}(\mathbb{R})$
since $\lambda e^{\lambda x} \in C^{\infty}(\mathbb{R})$

$= \int_{-a}^{a} H(x)(-e^{\lambda x}\varphi'(x))\,dx - \int_{-a}^{a} H(x)\lambda e^{\lambda x}\varphi(x)$

$= -\int_0^a 1\, e^{\lambda x}\varphi'(x)\,dx - \int_0^a 1\lambda e^{\lambda x}\varphi(x)\,dx =$

$= -e^{\lambda x}\varphi(x)\Big]_{x=0}^{x=a} + \int_0^a \lambda e^{\lambda x}\cancel{\varphi(x)\,dx}$

is zero since $\varphi \in C_0^{\infty}(-a,a)$

$- \int_0^a \lambda e^{\lambda x}\cancel{\varphi(x)\,dx} = + e^{\lambda 0}\varphi(0)$

$= \varphi(0) \Longleftrightarrow (\delta(x), \varphi(x))$

$\left(\frac{d}{dx} - \lambda\right)\left(H(x)e^{\lambda x}\right) = \delta(x)$

in the distributional sense