Solutions to Math 46 homework problems, Day 14

Exercise 2 page 257

$Lu = u'' + 4u$ $\qquad -(1u')' - 4u = -f$

$u(0) = u(\pi) = 0$ $\triangle$

Determine if there is a Green's function and solve the boundary value problem

$Lu = f$ subject to $\triangle$

We start by looking at whether 0 is an eigenvalue of L

$Lu - 0u = 0$ $\qquad \mathcal{X}$ = subspace of $C^2[0,\pi]$ formed by functions satisfying $\triangle$

$u'' + 4u = 0$

The general solution is

$u(t) = c_1 \cos 2t + c_2 \sin 2t$

$u(0) = 0 \Rightarrow c_1 = 0$

$u(\pi) = 0$ gives no conditions on c_2

Thus $\varphi(t) = \sin 2t$ is an eigenfunction of L corresponding to the 0 eigenvalue.

So there is <u>no</u> Green's function

Thus by theorem 4.23 the solution can be found only if $(f, \varphi)=0$ i.e. if $\int_0^{\pi} f(t)\sin 2t\,dt=0$ $\int_0^{\pi} f(t)\sin 2t\,dt=0$

We have to find $v(t)$ that satisfies $Lv=0$ and no required boundary conditions and is independent from φ. So we take $v(t)=\cos(2t)$

$$W(t)=\det\begin{pmatrix}\varphi & v\\ \varphi' & v'\end{pmatrix}=\det\begin{pmatrix}\cos 2t & \sin 2t\\ -2\sin 2t & 2\cos 2t\end{pmatrix}$$

$$=2 \qquad p(t)=1$$

By Theorem 4.23 we define

$$G(x,\xi)=-\frac{1}{p(\xi)W(\xi)}\Big(\varphi(x)v(\xi)H(\xi-x)+ \\ +\varphi(\xi)v(x)H(x-\xi)\Big)$$

$$=-\frac{1}{1\cdot 2}\Big(\sin(2x)\cos(2\xi)H(\xi-x)+ \\ +\sin(2\xi)\cos(2x)H(x-\xi)\Big)$$

$$u(x)=\underset{\substack{\uparrow\\ \text{arbitrary}\\ \text{constant}}}{c}\,\varphi(x)+\int_0^{\pi}G(x,\xi)f(\xi)\,d\xi$$

Exercise 7 page 258

By finding Green's function in two different ways, evaluate the sum $\sum_{n=1}^{\infty} \frac{\sin nx \sin n\xi}{n^2}$ $\quad 0 < x, \xi < \pi$

Solution By the computation on pages 256-257 for a nonsingular SLP

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}$$

where φ_n are eigenfunctions and λ_n are eigenvalues,

So in our case it would work if we find an SLP s.t.

$\varphi_n(x) = \sin(nx) \qquad 0 < x < \pi$

$\lambda_n = n^2$

We certainly know such SLP

it is
$$\left.\begin{aligned} -(1u')' + 0u &= 0 \\ u(0) = u(\pi) &= 0 \end{aligned}\right\}$$

it has eigenfunctions

$\varphi_n(x) = \sin(nx)$

$\lambda_n = n^2 \qquad n = 1, 2, 3, 4, 5$

page 4

As we can easily check 0 is not an eigen value.

$$Lu - 0u = 0 \text{ gives us}$$

Indeed

$$-u'' - 0u = 0$$
$$u(0) = u(\pi) = 0 \quad \triangle$$

$u(t) = At + B$ $\triangle$ implies that $u \equiv 0$ and thus 0 is <u>not</u> an eigen value of L.

Thus there is green's function

<u>we find $u_1(t)$ s.t.</u>

$$\left.\begin{array}{l} Lu_1 = 0 \\ u_1(0) = 0 \end{array}\right\} \text{ say we can take } u_1(t) = t$$

<u>We find $u_2(t)$ s.t.</u>

$$\left.\begin{array}{l} Lu_2 = 0 \\ u_2(\pi) = 0 \end{array}\right\} \text{ say we could take } u_2(t) = t - \pi$$

$$W = \det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} = \det\begin{pmatrix} t & t-\pi \\ 1 & 1 \end{pmatrix}$$

$$= \pi$$

By Theorem 4.19 we have

$$g(x,\xi) = -\frac{1}{\underbrace{p(\xi)}_{=1}\underbrace{w(\xi)}_{=\pi}}\Big(H(x-\xi)\,\xi\,(x-\pi) + H(\xi-x)\,x\,(\xi-\pi)\Big)$$

Thus we have our answer

$$\sum_{n=1}^{\infty} \frac{\sin(nx)\sin(n\xi)}{n^2} =$$

$$= -\frac{1}{\pi}\Big(H(x-\xi)\,\xi\,(x-\pi) + H(\xi-x)\,x\,(\xi-\pi)\Big)$$