Solutions to Math 46 homework problems Day 13

(3) page 257

Consider the boundary value problem

$u'' - 2xu' = f(x) \quad 0<x<1 \quad u(0)=u'(1)=0$

Find Green's function or explain why there is not one

Write the solution as the integral involving the Green's function

Solution This is not a SLP

multiply Both sides by e^{-x^2}

$$\left(-e^{-x^2}u'' + 2xe^{-x^2}u'\right) + 0u = f(x)e^{-x^2}$$

$$\parallel$$

$$-(\underbrace{e^{-x^2}}_{p(x)}u')' + \underbrace{0}_{q(x)}u = \underbrace{f(x)e^{-x^2}}_{\hat{f}(x)}$$

$p(x)$ is never zero so this is a nonsingular SLP and the Green's function method can be attempted

$$\left.\begin{array}{l} Lu = -(e^{-x^2}u')' + 0u \\ u(0)=0 \\ u'(1)=0 \end{array}\right\}$$

Is zero the eigen value of L
i.e. is there a nontrivial solution of

$$\left.\begin{array}{l} \left(-(e^{-x^2}u')' + 0u\right) - 0u = 0 \\ u(0)=0 \\ u'(1)=0 \end{array}\right\}$$

$$-\cancel{e^{-x^2}}u'' + 2x\cancel{e^{-x^2}}u' = 0$$

$$-u'' + 2xu' = 0$$

Put $v = u'$

$$-v' + 2xv = 0$$

$$v' - 2xv = 0$$ multiply both sides by e^{-x^2}

$$\underline{e^{-x^2}v'(x) - 2xe^{-x^2}v(x)} = 0$$

$$\| $$

$$(e^{-x^2}v(x))'$$

$$\Rightarrow \quad e^{-x^2}v(x) = A$$ some constant

$v(x) = Ae^{x^2}$

$v(x) = u'(x) \Rightarrow$

$u(x) = \int_0^x Ae^{t^2} dt + B$ (*)

$u(0)=0 \Rightarrow 0 = \int_0^0 Ae^{t^2} dt + B \Rightarrow B = 0$

$u'(x) = Ae^{x^2}$ $u'(1) = 0 \Rightarrow A = 0$

$\Rightarrow u(x) = \int_0^x 0 e^{t^2} dt + 0 \equiv 0$ and

0 is not an eigenvalue.

Now we have to find

$u_1(x)$ satisfying

$\left.\begin{array}{l} Lu_1 = 0 \\ u_1(0) = 0 \end{array}\right\}$

$u_2(x)$ satisfying

$\left.\begin{array}{l} u_2'(1) = 0 \\ Lu_2 = 0 \end{array}\right\}$

By (*) $u_1(x) = \int_0^x Ae^{t^2} dt + B$

$u_1(0) = 0 \Rightarrow B = 0$

Take $u_1(x) = \int_0^x e^{t^2} dt$

By (*) $u_2(x) = \int_0^x Ae^{t^2} dt + B$

$u_2'(x) = Ae^{x^2}$ $u_2'(1) = 0 \Rightarrow A = 0 \Rightarrow u_2 = B$

Choose $u_2 = 1$

Thus $u_1(x) = \int_0^x e^{t^2}\,dt$

$u_2(x) = 1$

$$W(u_1, u_2) = \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} =$$

$$= \det \begin{pmatrix} \int_0^x e^{t^2}\,dt & 1 \\ e^{x^2} & 0 \end{pmatrix} = -e^{x^2}$$

$$g(x,\xi) = -\frac{1}{p(\xi)W(\xi)} \begin{pmatrix} H(x-\xi)\, u_1(\xi)\, u_2(x) \\ + H(\xi - x)\, u_1(x)\, u_2(\xi) \end{pmatrix}$$

$$= -\frac{1}{e^{-x^2}(-e^{x^2})} \begin{pmatrix} H(x-\xi) \int_0^\xi e^{t^2}\,dt\; 1 - \\ - H(\xi - x) \int_0^x e^{t^2}\,dt \cdot 1 \end{pmatrix}$$

$$\| \, 1$$

$$u(x,\xi) = \int_0^1 g(x,\xi)\,\tilde{f}(\xi)\,d\xi =$$

$$= \int_0^1 g(x,\xi)\, f(\xi)\, e^{-\xi^2}\, d\xi$$

Exercise 4 page 257

Consider the boundary value problem

$$u'' + u' - 2u = f(x) \quad 0 \leq x \leq 1$$

$$u(0) = u'(1) = 0$$

Find Green's function or explain why there is not one.
Find the solution in terms of the Green's function

Solution This is not an SLP

But it becomes an SLP if you multiply this by $-\tilde{q}(x)$ so that

$$-(\tilde{q}(x)u'' + \tilde{q}(x)u') + 2\tilde{q}(x)u = \underbrace{-f(x)\tilde{q}(x)}_{\hat{f}(x)}$$

becomes an SLP

$$(\tilde{q}(x)u'' + \tilde{q}(x)u')$$

$$\| \leftarrow \text{indeed true for}$$

$$(p(x)u')' \quad \tilde{q}(x) = e^x$$

$$p(x) = e^x$$

So multiply both sides of the equation by $-e^x$

$$-e^x u'' - e^x u' + 2e^x u = \underbrace{-f(x)e^x}_{\hat{f}(x)}$$

$$\underbrace{-(e^x u')'}_{} + 2e^x u = \hat{f}(x)$$

$$-(\underbrace{e^x}_{p(x)} u')' + \underbrace{2e^x}_{q(x)} u = \hat{f}(x)$$

Now $p(x)$ is never zero,
$p, q, \hat{f}$ continuous so this is a
nonsingular SLP.

$$\left.\begin{array}{l} Lu = -(e^x u')' + 2e^x u \\ u(0) = u'(1) = 0 \quad \Delta \end{array}\right\}$$

is 0 an eigenvalue of L
in the subspace $\mathcal{H} \subset C^2[0,1]$
formed by functions satisfying Δ

$$-(e^x u')' + 2e^x u = 0$$

$$-\cancel{e^x} u'' - \cancel{e^x} u' + 2\cancel{e^x} u = 0$$

$$u'' + u' - 2u = 0$$

$$r^2 + r - 2 = 0$$

$$r_{1,2} = \frac{-1 \pm \sqrt{9}}{2} \qquad r_1 = -2$$

$$r_2 = 1$$

The general solution is

$$u(x) = c_1 e^{-2x} + c_2 e^{x}$$

$$u(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$\Rightarrow u(x) = c_1 (e^{-2x} - e^{x})$$

$$u'(x) = c_1 (\underbrace{-2e^{-2x} - e^{x}}_{\neq 0})$$

$$u'(1) = 0 \Rightarrow c_1 = 0 \Rightarrow u \equiv 0$$

and 0 is not an eigenvalue of L on $\mathcal{H}$

Now we find Green's function

$$\left.\begin{array}{l} Lu_1 = 0 \\ u_1(0) = 0 \end{array}\right\} \quad \begin{array}{l} u_1(x) = c_1 e^{-2x} + c_2 e^{x} \\ u_1(0) = 0 \Rightarrow c_1 = -c_2 \end{array}$$

Take $u_1(x) = e^{-2x} - e^{x}$

$$\left.\begin{array}{l} Lu_2 = 0 \\ u_2'(1) = 0 \end{array}\right\} \quad \begin{array}{l} u_2(x) = c_1 e^{-2x} + c_2 e^{x} \\ u_2'(x) = -2c_1 e^{-2x} + c_2 e^{x} \end{array}$$

$$u_2'(1) = -2c_1 e^{-2} + c_2 e^{1} = 0$$

$$c_2 = \frac{2c_1 e^{-2}}{e} = 2c_1 e^{-3}$$

choose $c_1 = 1$ $c_2 = 2e^{-3}$

$$u_2(x) = e^{-2x} + 2e^{-3} e^{x}$$

$$W(u_1,u_2) = \det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} =$$

$$= \det\begin{pmatrix} e^{-2x} - e^{x} & e^{-2x} + 2e^{x-3} \\ -2e^{-2x} - e^{x} & -2e^{-2x} + 2e^{x-3} \end{pmatrix} =$$

$$= -\cancel{2e^{-4x}} + 2e^{-x-3} + 2e^{-x} - 2e^{-3} +$$

$$+ \cancel{2e^{-4x}} + 4e^{-x-3} + e^{-x} + 2e^{2x-3} =$$

$$= 6e^{-x-3} + 2e^{2x-3} + 3e^{-x} - 2e^{-3}$$

$$g(x,\zeta) = -\frac{1}{p(\zeta)W(\zeta)}\left(H(x-\zeta)u_1(\zeta)u_2(x) + H(\zeta-x)u_1(x)u_2(\zeta)\right)$$

so we get

$$g(x,\zeta) = \frac{-1}{e^{\zeta}(6e^{-\zeta-3} + 2e^{2\zeta-3} + 3e^{-\zeta} - 2e^{-3})} \cdot$$

$$\cdot \left(H(x-\zeta)(e^{-2\zeta} - e^{\zeta})(e^{-2x} + 2e^{x-3}) + H(\zeta-x)(e^{-2x} - e^{x})(e^{-2\zeta} + 2e^{\zeta-3}\right)$$

$$Uf(x) = \int_0^1 g(x,\zeta)\tilde{f}(\zeta)\,d\zeta = \int_0^1 g(x,\zeta)(-f(\zeta)e^{\zeta})\,d\zeta$$

Exercises problem 257

Use the method of Green's function to solve the problem

$$\left.\begin{array}{l} -(K(x)u')' = f(x) \quad 0<x<1 \\ u(0)=u(1)=0 \\ K(x)>0 \end{array}\right\}$$

Solution This is a nonsingular SLP. Let us check that 0 is not an eigenvalue of

$L(u) = -(K u')'$ on

$\mathfrak{X}$ - formed by functions u satisfying $u(0)=u(1)=0$

$\cap$

$C^2[0,1]$

$Lu - 0u = 0 \Rightarrow Lu = 0$

$$-(K(x)u')' = 0 \Rightarrow -K(x)\,u'(x) = \underbrace{A}_{\text{some constant}}$$

$$\Rightarrow u'(x) = \frac{-A}{K(x)} \Rightarrow u(x) = \int_0^x \frac{-A}{K(t)}\,dt + B$$

$$u(0)=0 \Rightarrow \int_0^0 \frac{-A}{K(t)}\,dt + B = 0 \Rightarrow B = 0$$

$$u(1)=0 \Rightarrow \underbrace{\int_0^1 -\frac{A}{K(t)}\,dt}_{\text{would be equal to zero only if } A=0} + 0 = 0$$

zero only if $A=0 \Rightarrow u \equiv 0$

So 0 is not an eigenvalue of L on $\mathcal{H}$

Now we have to find

u_1 s.t $\left.\begin{array}{l} Lu_1 = 0 \\ u_1(0) = 0 \end{array}\right\}$

$Lu_1 = 0 \Rightarrow u_1(x) = \int_0^x \frac{-A}{k(t)}\,dt + B$

$u_1(0) = 0 \Rightarrow B = 0$ Choose

$u_1(x) = \int_0^x \frac{1}{k(t)}\,dt$

We have to find u_2 s.t

$Lu_2 = 0$
$u_2(1) = 0$

$Lu_2 = 0 \Rightarrow u_2(x) = \int_0^x \frac{-A}{k(t)}\,dt + B$

$u_2(1) = 0 \Rightarrow B = \int_0^1 \frac{A}{k(t)}\,dt$

Choose $A = -1$

$u_2(x) = \int_0^x \frac{1}{k(t)}\,dt - \int_0^1 \frac{1}{k(t)}\,dt$

$W = \det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} =$

$$\det \begin{pmatrix} \int_0^x \frac{1}{k(t)}\,dt & \int_0^x \frac{1}{k(t)}\,dt - \int_0^1 \frac{1}{k(t)}\,dt \\ \frac{1}{k(x)} & \frac{1}{k(x)} \end{pmatrix} =$$

$$= \frac{1}{k(x)} \int_0^1 \frac{1}{k(t)}\,dt$$

$$g(x,\xi) = -\frac{1}{p(\xi)\,w(\xi)} \begin{pmatrix} H(x-\xi)\,u_1(\xi)\,u_2(x) + \\ + H(\xi-x)\,u_1(x)\,u_2(\xi) \end{pmatrix}$$

$$= -\frac{1}{\underbrace{\cancel{k(\xi)}}_{p(\xi)} \cdot \underbrace{\cancel{\frac{1}{k(\xi)}} \int_0^1 \frac{1}{k(t)}\,dt}_{w(\xi)}} \cdot$$

$$\cdot \begin{pmatrix} H(x-\xi) \int_0^\xi \frac{1}{k(t)}\,dt \left(\int_0^x \frac{1}{k(t)}\,dt - \int_0^1 \frac{1}{k(t)}\,dt \right) \\ + H(\xi-x) \int_0^x \frac{1}{k(t)}\,dt \left(\int_0^\xi \frac{1}{k(t)}\,dt - \int_0^1 \frac{1}{k(t)}\,dt \right) \end{pmatrix}$$

$$u(x) = \int_0^1 g(x,\xi)\, f(\xi)\, d\xi$$