Solutions to Math 46 homework problems Day 12

Exercise 6 page 244

$$u(t)=1+\int_0^t s\ln\left(\frac{s}{t}\right)u(s)\,ds \qquad (*)$$

Reformulate this as an initial value problem

Take a derivative of both sides

$$u'(t)= \underbrace{t\ln\left(\frac{t}{t}\right)u(t)}_{=0} + \int_0^t s\,\underbrace{\frac{1}{\left(\frac{s}{t}\right)}\left(-\frac{s}{t^2}\right)}\,u(s)\,ds$$

$$\int_0^t \frac{-s^2 t}{\underbrace{s\,t^2}}\,u(s)\,ds$$

$$u'(t) \overset{(**)}{=} \int_0^t -\frac{s}{t}\,u(s)\,ds$$

$$u'(t) = \int_0^t -\frac{s}{t}\,u(s)\,ds = -\frac{1}{t}\int_0^t s u(s)\,ds$$

$$t u'(t) = -\int_0^t s u(s)\,ds$$

Take a derivative again

$$t u''(t) + u'(t) = -t u(t)$$

From (*) we have $u(0)=1$

From (**) we have $u'(0)=0$

Thus the answer is

$$\left.\begin{array}{l} t u''(t)+u'(t) = -t u(t) \\ u(0)=1 \qquad u'(0)=0 \end{array}\right\}$$

Exercise 8 page 244.

$$u(t) = \underset{\uparrow \atop f}{t} + \mu \underbrace{\int_0^t (t-s)\,u(s)\,ds}_{Ku}$$

Thus $\quad s_0(t) = f = t$

$$s_1(t) = \underset{\overset{\parallel}{t}}{f(t)} + \mu K(s_0) =$$

$$= t + \mu \int_0^t (t-s)\,s\,ds = t + \mu\left(\frac{ts^3}{2} - \frac{s^3}{3}\right)\Bigg]_{s=0}^{s=t}$$

$$= t + \mu\left(\frac{t^3}{2} - \frac{t^3}{3}\right) = t + \mu\frac{t^3}{6}$$

$$s_2 = \underset{\overset{\parallel}{t}}{f(t)} + \mu \int_0^t (t-s)\left(s + \mu\frac{s^3}{6}\right)ds =$$

$$= t + \mu \int_0^t ts - s^2 + \frac{t\mu s^3}{6} - \frac{\mu s^4}{6}\,ds =$$

$$= t + \mu\left(\frac{ts^3}{2} - \frac{s^3}{3} + \frac{t\mu s^4}{24} - \frac{\mu s^5}{30}\right)\Bigg]_{s=0}^{s=t}$$

$$= t + \mu\left(\frac{t^3}{2} - \frac{t^3}{3} + \mu\frac{t^5}{24} - \frac{\mu t^5}{30}\right) =$$

$$= t + \mu\frac{t^3}{6} - \mu^2\frac{t^5}{120}$$

Exercise 10 page 245

Reformulate the initial value problem $u'' - \lambda u = f(x) \quad x > 0$

$$u(0) = 1$$
$$u'(0) = 0$$

as a Volterra integral equation

$$u''(s) = \lambda u(s) + f(s) \quad \forall s > 0$$

integrate over s

$$\Rightarrow \quad \int_0^r 1\, u''(s)\,ds = \lambda \int_0^r u(s)\,ds + \int_0^r f(s)\,ds$$

holds $\forall r > 0$

$u''(s) = (u')'$

$$1\, u'(s) \Big]_{s=0}^{s=r} = u'(r) - \underbrace{u'(0)}_{=0}$$

$$\Rightarrow \quad u'(r) = \lambda \int_0^r u(s)\,ds + \int_0^r f(s)\,ds$$

Integrate over r for all r from $r = 0$ to $r = x$

$$\int_0^x u'(r)\,dr = \lambda \int_0^x \int_0^r u(s)\,ds + \int_0^x \int_0^r f(s)\,ds$$

$$= u(x) - \underbrace{u(0)}_{=1}$$

$$\Rightarrow u(x) = 1 + \int_0^x \int_0^r f(s)\,ds + \lambda \underbrace{\int_0^x \int_0^r u(s)\,ds}_{\parallel}$$

$$\lambda \int_0^x u(s)(x-s)\,ds$$

$$u(x) = \underbrace{1 + \int_0^x \int_0^r f(s)\,ds}_{\tilde{f}(x)} + \lambda \int_0^x \underbrace{(x-s)}_{k(x,s)} u(s)\,ds$$

Is the desired Volterra equation