Solutions for Math 46 homework problems Day 11

Exercise 13 part b page 245

$$u(x) = b + \lambda \int_0^1 (u(y))^2 \, dy$$

The right handside does not depend on x, so the left hand side also does not depend on x $\Rightarrow$ $u(x) \equiv A$ ← some constant

Now if $\lambda = 0$ then we get

$A = b + 0 \Rightarrow u(x) \equiv b$ is the solution

If $\lambda \neq 0$ we get

$$A = b + \lambda \int_0^1 A^2 \, dy = b + \lambda A^2$$

$$\lambda A^2 - A + b = 0$$

$$A_{1,2} = \frac{1 \pm \sqrt{1 - 4\lambda b}}{2\lambda} \quad (*)$$

Thus if $\lambda \neq 0$ is such that $1 - 4\lambda b > 0$ then we have two solutions $u(x) = A_{1,2}$ given by $*$

this is

if $1=4\lambda b$ then we have one solution $u(x)=\frac{1}{2\lambda}$

if $1-4\lambda b<0$ then we have no real valued solution functions $u(x)$

Exercise 15 page 245

Solve the Fredholm equation

$$\int_0^1 k(x,y)u(y)\,dy - \lambda u(x) = x$$

using eigenfunction expansions where

$$k(x,y)=\begin{cases} x(1-y) & \text{if } x<y \\ y(1-x) & \text{if } x>y \end{cases}$$

Solution The kernel is symmetric and real valued. It is continuous since the only possible set of discontinuity points is along the line $y=x$ where $x(1-y)$ and $y(1-x)$ match and become $x(1-x)$.

Thus we can use Hilbert-Schmidt Theorem.

We search for eigen functions and eigen values

$$Ku = \int_0^1 k(x,y)u(y)\,dy$$

$$Ku = \lambda u$$

$$Ku(x) = \int_0^x y(1-x)u(y)\,dy + \int_x^1 x(1-y)u(y)\,dy$$

Leibnitz rule

cancel

$$\text{So } (Ku(x))' = x(1-x)u(x)\frac{dx}{dx} - \underbrace{0(1-x)u(0)\frac{d0}{dx}}_{=0} + \int_0^x \underbrace{\frac{d}{dx}(y(1-x)u(y))}_{=-y\,u(y)}dy$$

$$+ \underbrace{x(1-1)u(1)\frac{d1}{dx}}_{=0} - x(1-x)u(x)\underbrace{\frac{dx}{dx}}_{=1}$$

$$+ \int_x^1 \underbrace{\frac{d}{dx}(x(1-y)u(y))}_{=1(1-y)u(y)}dy =$$

$$= \int_0^x -y u(y)\,dy + \int_x^1 (1-y)u(y)\,dy$$

Thus $(Ku(x))' = \int_0^x -y u(y)\,dy + \int_x^1 (1-y)u(y)\,dy$

$(Ku(x))'' = -xu(x) - (1-x)u(x) = -u(x)$

Thus $(Ku(x))'' = -u(x)$

but $Ku = \lambda u \Rightarrow (Ku)'' = \lambda u''$

Thus we get $\lambda u''(x) = -u(x)$

If $\lambda = 0$ it has no nontrivial solutions

$Ku(0) = \lambda u(0)$ so WLOG $\lambda \neq 0$

$$\| $$

$$\int_0^0 y(1-0)u(y)\,dy + \int_0^1 0(1-y)u(y)\,dy = 0$$

$Ku(1) = \lambda u(1)$

$$\| $$

$$\int_0^1 y\underbrace{(1-1)}_{=0}u(y)\,dy + \int_1^1 1(1-y)u(y)\,dy = 0$$

$\Rightarrow \quad u(1) = 0$

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Thus we get a Sturm Liouville problem

$$\left.\begin{aligned} u'' + \frac{1}{\lambda} u &= 0 \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}\right\}$$

$\tilde{\lambda}$

As we already know the eigenvalues a $\tilde{\lambda} = \left(\frac{\pi n}{1}\right)^2$ $n = 1, 2, 3, \ldots$

The eigenfunctions are

$$u_n(x) = \sin \frac{\pi n x}{1}$$

These are not of norm 1

$$(u_n, u_n) = \int_0^1 \sin^2(\pi n x)\, dx = \int_0^1 \frac{1}{2} - \frac{\cos(2\pi n x)}{2} dx = \frac{1}{2}$$

$$1 - 2\sin^2\alpha = \cos 2\alpha$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\| u_n \| = \frac{1}{\sqrt{2}}$$

Note that Hilbert-Schmidt Theorem method works even when eigenfunctions are not of norm 1 page 6

Thus for the Fredholm equation we have

$$\lambda_n = \frac{1}{\tilde{\lambda}} = \frac{1}{(\pi n)^2} \leftarrow \text{eigenvalues.}$$

$$u_n(x) = \sin(\pi n x) \leftarrow \text{eigenfunctions}$$

$n = 1, 2, 3, 4, \ldots$

Let us find the presentation

$$x = \sum_{n=1}^{\infty} f_n u_n(x)$$

As we know from Fourier series discussions $f_n = \frac{(x, u_n)}{\underbrace{(u_n, u_n)}_{= \frac{1}{2}}} = 2(x, u_n)$

$$(f, u_n) = \int_0^1 x \underbrace{\sin(\pi n x)}_{= \left(-\frac{1}{\pi n}\cos(\pi n x)\right)'} dx =$$

$$= -\frac{x}{\pi n}\cos(\pi n x)\Big]_{x=0}^{x=1} + \int_0^1 \frac{\cos(\pi n x)}{\pi n}(x)' dx$$

$$= -\frac{1}{\pi n}\underbrace{\cos(\pi n)}_{= (-1)^n} + \underbrace{\frac{\sin(\pi n x)}{(\pi n)^2}\Big]_{x=0}^{x=1}}_{= 0} =$$

$$= \frac{(-1)^{n+1}}{\pi n}$$

Assume $u(x) = \sum_{n=1}^{\infty} \alpha_n u_n(x)$

$$K(u(x)) - \lambda(\sum_{n}$$

$$K\left(\sum_{n=1}^{\infty} \alpha_n u_n(x)\right) - \lambda\left(\sum_{n=1}^{\infty} \alpha_n u_n(x)\right)$$

$$\| $$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} u_n(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n K(u_n(x)) - \sum_{n=1}^{\infty} \alpha_n \lambda u_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} u_n(x)$$

$$\sum_{n=1}^{\infty} \alpha_n \underbrace{\lambda_n}_{= \frac{1}{(\pi n)^2}} u_n(x) - \sum_{n=1}^{\infty} \alpha_n \lambda u_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} u_n(x)$$

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n}{(\pi n)^2} - \alpha_n \lambda\right) u_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} u_n(x)$$

Compare the coefficients in front of $u_n(x)$

$$\alpha_n \left(\frac{1}{(\pi n)^2} - \lambda\right) = \frac{(-1)^{n+1}}{\pi n}$$

$$\alpha_n = \frac{(-1)^{n+1}}{\frac{1}{(\pi n)^2} - \lambda} \quad \text{if } \lambda \neq \frac{1}{(\pi n)^2}$$

Thus if $\lambda \neq \frac{1}{(\pi n)^2}$ $\forall n$ then the solution is

$$u(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\frac{1}{(\pi n)^2} - \lambda} \sin(\pi n x)$$

if $\lambda = \frac{1}{(\pi \tilde{n})^2}$ then there is no solution since the corresponding coefficient for the function x is $\frac{(-1)^{n+1}}{\pi \tilde{n}} \neq 0$

Exercise 18 page 246

Solve the integral equation

$$\int_0^1 e^{x+y} u(y)\,dy - \lambda u(x) = f(x)$$

by considering all cases

$$e^{x+y} = \underset{\alpha_1(x)}{\underset{\parallel}{e^x}}\ \underset{\beta_1(y)}{\underset{\parallel}{e^y}}$$ So this is a

Fredholm equation with separable kernel. The matrix A is 1×1

$$(\beta_1, \alpha_1) = \int_0^1 e^x e^x\,dx = \frac{1}{2} e^{2x} \Big]_{x=0}^{x=1} =$$

$$= \frac{1}{2}(e^2 - 1)$$

The column vector F is $(\beta_1, f) =$

$$= \int_0^1 e^x f(x)\,dx$$

Thus equation 4.31 becomes

$$\sum_{j=1}^{n} \alpha_j(x \qquad (A - \lambda I)\vec{c} = \vec{F}$$

$$\left(\frac{1}{2}(e^2-1) - \lambda 1\right) c = \int_0^1 e^x f(x)\,dx$$

The eigenvalue of A clearly is $\tilde{\lambda} = \frac{1}{2}(e^2-1)$

Assume that $\lambda \neq \tilde{\lambda}$ and $\lambda \neq 0$

Then, as we know

$$c = \frac{\int_0^1 e^x f(x)\,dx}{\frac{1}{2}(e^2-1)-\lambda} \quad \text{and}$$

The solution is

$$u(x) = \frac{1}{\lambda}\left(-f(x) + c\,\alpha_1(x)\right) =$$

$$= \frac{1}{\lambda}\left(-f(x) + \frac{\int_0^1 e^y f(y)\,dy}{\frac{1}{2}(e^2-1)-\lambda}\, e^x\right)$$

If $\tilde{\lambda} = \frac{1}{2}(e^2-1)$ then the solution exists only if $\vec{F}$ is in the range of $\underline{A - \tilde{\lambda} I} = 0$

zero operator

i.e. only if $\vec{F} = 0$ i.e. only if $\int_0^1 e^x f(x)\,dx = 0$ in this case it is $u(x) = \frac{1}{\tilde{\lambda}}\left(-f(x) + \tilde{c}\, e^x\right)$

any constant

If $\lambda = 0$ then the equation becomes

$$\int_0^1 e^x e^y u(y)\,dy - 0u(x) = f(x)$$

$$\Updownarrow$$

$$e^x \underbrace{\int_0^1 e^y u(y)\,dy}_{\uparrow \text{ some constant}} = f(x)$$

So the solution could exist only for $f(x) = De^x$ To find a solution in this case take any $\tilde{u}(y)$ s.t. $\int_0^1 e^y u(y)\,dy \neq 0$ multiply $\tilde{u}(y)$ by a constant to make this integral D. This would be a particular solution Plus you can add any solution to a homogeneous problem

$$e^x \underbrace{\int_0^1 e^y u(y)\,dy}_{E} = 0$$

Thus you can look at and $\hat{u} = \tilde{u} + u$ where u satisfies $\int_0^1 e^y u(y)\,dy = 0$