Solutions to homework problems
Day 10

Exercise 4.b page 244
Find the eigenvalues and orthonormal eigenfunctions associated with the integral operator

$$Ku = \int_0^1 \min(x,y)u(y)\,dy =$$

$$\min(x,y) = \begin{cases} x & \text{if } x \le y \\ y & \text{if } x \ge y \end{cases}$$

$$= \int_0^x y u(y)\,dy + \int_x^1 x u(y)\,dy \qquad (*)$$

We are trying to solve

$$Ku(x) = \lambda u(x)$$

Take the $\frac{d}{dx}$ derivative of both sides

$$\lambda u'(x) = \frac{d}{dx}\left(\int_0^x y u(y)\,dy + \int_x^1 x u(y)\,dy\right)$$

$$\underset{\uparrow}{=} \cancel{x u(x)} - \cancel{x u(x)} + \int_x^1 \underbrace{\frac{\partial}{\partial x}(x u(y))}_{u(y)}\,dy$$

Leibnitz formula

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Thus $\lambda u'(x) = \int_x^1 u(y)\,dy \quad (**')$

Take the derivative again

$$\lambda u''(x) = -u(x)$$

$$\lambda u''(x) + u(x) = 0 \quad (\Delta)$$

From $(*)$ we can figure out $u(0)$

$$(Ku)(0) = \lambda u(0) = \int_0^0 y u(y)\,dy + \int_0^1 0\,u(y)\,dy = 0$$

$$(Ku)(\phi) = \lambda u(1)$$

From $(**')$ we get

$$\lambda u'(\phi) = \int_1^1 u(y)\,dy = 0$$

Now if λ is $=0$ then (Δ) has only the trivial solution.

Thus $\lambda \neq 0$ and we have

$$\left.\begin{array}{l} \lambda u'' + u = 0 \\ u(0) = 0 \\ u'(1) = 0 \end{array}\right\} \Leftrightarrow u'' + \frac{1}{\lambda} u = 0$$

(A) $\frac{1}{\lambda} = \mu^2 > 0 \Rightarrow$ the general solution is $u(x) = c_1\cos(\mu x) + c_2 \sin(\mu x)$

$u(0)=0 \Rightarrow c_1 = 0 \Rightarrow$

$u(x) = c_2 \sin(\mu x)$

$u'(x) = c_2 \underset{\neq 0}{\mu} \cos(\mu x)$

since

$u'(1) = 0 \Rightarrow \mu = \frac{\pi}{2} + \pi n$

$n = 0, 1, 2, 3 \ldots$

Thus we get eigenvalues $\lambda = \frac{1}{\mu^2}$

$\lambda_n = \frac{1}{\left(\frac{\pi}{2} + \pi n\right)^2} \qquad n = 0, 1, 2, 3, 4 \ldots$

The corresponding eigenfunctions are $u_n(x) = \cos\left(\left(\frac{\pi}{2} + \pi n\right)x\right)$

If $\frac{1}{\lambda} = -\mu^2 < 0$ then the general solution is

$u(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$u(0) = 0 \Rightarrow c_1 = -c_2 \Rightarrow$

$u(x) = c_1\left(e^{\mu x} - e^{-\mu x}\right)$

$u'(x) = c_1\left(\mu e^{\mu x} + \mu e^{-\mu x}\right) = c_1 \mu\left(e^{\mu x} + e^{-\mu x}\right)$

$u'(1) = 0 \Rightarrow c_1 = 0$ but then this is not an eigen function

What are the lengths (norms) of

$u_n(x) = \cos\left(\left(\frac{\pi}{2}+\pi n\right)x\right)$

$$\|u_n(x)\|^2 = \int_0^1 \cos^2\left(\left(\frac{\pi}{2}+\pi n\right)x\right)dx =$$

$$2\cos^2\alpha - 1 = \cos 2\alpha \Rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= \int_0^1 \frac{1}{2} + \frac{\cos\left(2\left(\frac{\pi}{2}+\pi n\right)x\right)}{2}\,dx =$$

$$= \frac{1}{2} + \frac{1}{2}\int_0^1 \cos((\pi+2\pi n)x)\,dx =$$

$$= \frac{1}{2} + \frac{1}{2}\,\frac{1}{\pi+2\pi n}\underbrace{\sin((\pi+2\pi n)x)\Big]_{x=0}^{x=1}}_{=0} = \frac{1}{2}$$

$$\Rightarrow \|u_n(x)\| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Thus the orthonormal set of eigenvalues is

$$\tilde{u}_n(x) = \sqrt{2}\cos\left(\left(\frac{\pi}{2}+\pi n\right)x\right)$$

$n = 0, 1, 2, 3, 4, \ldots$

corresponding to the eigenvalues

$$\lambda_n = \frac{1}{\left(\frac{\pi}{2}+\pi n\right)^2} \quad n = 0, 1, 2, 3, 4, \ldots$$

Exercise 13 part a page 245

Discuss the solvability of
$u(x) = f(x) + \lambda \int_0^{1/2} u(y)\,dy$ and
solve it if possible.

If $\lambda = 0$ then the equation becomes $u(x) = f(x)$ so we found $u(x)$. Otherwise we rewrite it as

$$\int_0^{1/2} 1\, u(y)\,dy - \underbrace{\frac{1}{\lambda}}_{\tilde{\lambda}} u(x) = \underbrace{-\frac{1}{\lambda} f(x)}_{\overset{\text{def}}{=} \tilde{f}(x)}$$

So we get

$$\int_0^{1/2} 1 \cdot 1\, u(y)\,dy - \tilde{\lambda} u(x) = \tilde{f}(x)$$

This is an equation with separable kernel

$\alpha_1(x) = 1$
$\beta_1(y) = 1$

The matrix A is a 1×1 matrix (β_1, α_1) is its only entry

$$(\beta_1, \alpha_1) = \int_0^{1/2} 1 \cdot 1\,dx = \frac{1}{2}$$

By Formulas 4.33 and 4.34 we have

$$(A-\tilde{\lambda} I)\vec{c} = \vec{F} \quad \text{where } F=(\tilde{f},\beta_1)$$

$$u(x) = \frac{1}{\tilde{\lambda}}\left(-\tilde{f}(x) + \sum_{j=1}^{n} \alpha_j(x)c_j\right) \qquad \int_0^{\frac{1}{2}} \tilde{f}(x)dx$$

This gives us

$$\left(\frac{1}{2} - \tilde{\lambda}\, 1\right) c = \int_0^{\frac{1}{2}} \tilde{f}(x)dx = \int_0^{\frac{1}{2}} -\frac{1}{\lambda} f(x)dx$$

if $\tilde{\lambda} = \frac{1}{2}$ ← $\lambda = 2$ then the solution exists only when $\int_0^{\frac{1}{2}} -\frac{1}{\lambda} f(x)dx = 0$ $\Leftrightarrow \int_0^{\frac{1}{2}} f(x)dx$. In this case c can be taken to be any constant

In this case

$$u(x) = \frac{1}{\left(\frac{1}{2}\right)}\left(-\frac{1}{2} f(x) + \cdot c\right)$$

arbitrary constant

is a solution for $\lambda = 2$ when $\int_0^{\frac{1}{2}} f(x)dx = 0$

for $\lambda = 2$ when $\int_0^{\frac{1}{2}} f(x)dx \neq 0$ there are no solutions when $\tilde{\lambda} \neq \frac{1}{2}$ (i.e. when $\lambda \neq 2$)

we have $\left(\frac{1}{2} - \frac{1}{\lambda}\right) c = -\frac{1}{\lambda} \int_0^{\frac{1}{2}} f(x)\,dx$

$$\Rightarrow c = -\frac{2\cancel{\lambda}}{\lambda - 2} \frac{1}{\cancel{\lambda}} \int_0^{\frac{1}{2}} f(x)\,dx = -\frac{2}{\lambda-2} \int_0^{\frac{1}{2}} f(x)\,dx$$

then

$$u(x) = \frac{1}{\left(\frac{1}{\lambda}\right)} \left(-\left(-\frac{1}{\lambda} f(x) \right) + c\,1 \right) =$$

$$= \lambda \left(\frac{1}{\lambda} f(x) + \frac{-2}{\lambda - 2} \int_0^{\frac{1}{2}} f(x)\,dx \right)$$

Exercise 16 page 245

Replace $\sin(xy)$ by the first two terms in its power series expansion to get an approximate solution to $u(x)=x^2+\int_0^1 \sin(xy)u(y)dy$

$$f(x,y)\approx f(0,0)+\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial y}(0,0)y+$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x\partial y}(0,0)xy+\frac{1}{2}\frac{\partial^2 f}{\partial y\partial x}(0,0)yx+$$
$$+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2$$

In our case we get

$$\sin(xy)\approx \sin(0)+0\cos(0\cdot 0)x+0\cos(0\cdot 0)y$$
$$+\frac{1}{2}(\cos 0-0\sin 0)xy+\frac{1}{2}(\cos 0-0\sin 0)xy$$
$$+\frac{1}{2}(-0\sin 0)x^2+\frac{1}{2}(-0\sin 0)y^2=xy$$

$$\frac{\partial f}{\partial x}=y\cos(xy)$$
$$\frac{\partial^2 f}{\partial x^2}=-y^2\sin(xy) \qquad \frac{\partial^2 f}{\partial x\partial y}=\cos(xy)-xy\sin(xy)$$
$$\frac{\partial f}{\partial y}=x\cos(xy) \qquad \frac{\partial^2 f}{\partial y^2}=-x^2\sin(xy)$$

Thus we search for the solution of

$$u(x) = x^2 + \int_0^1 xy\, u(y)\,dy \iff$$

$$\int_0^1 \underset{\substack{\uparrow \\ \text{separable} \\ \text{kernel}}}{xy}\, u(y)\,dy - \underset{\lambda}{1}\, u(x) = \underbrace{-x^2}_{f(x)} \quad (*)$$

The hope is that the solution of (*) will be close to the solution of the original equation

We have $\alpha_1(x) = x$
$\beta_1(y) = y$

A with the only entry $(\beta_1, \alpha_1) = \int_0^1 x x\,dx = \frac{1}{3}$

$\uparrow$ 1x1-matrix

F is a column of size one with the only entry $(\beta_1, f) = \int_0^1 x(-x^2)\,dx =$

$= -\frac{1}{4}$

The equation 4.33 becomes

$(A - \lambda I)\vec{c} = \vec{F}$ $\quad \left(\frac{1}{3} - 1\cdot 1\right) c = -\frac{1}{4}$

$-\frac{2}{3} c = -\frac{1}{4} \qquad c = \frac{3}{8}$

equation 4.34 becomes

$$u(x) = \frac{1}{\lambda}\left(-f(x) + \sum_{j=1}^{n} \alpha_j(x) c_j\right)$$

$$u(x) = \frac{1}{1}\left(-(-x^2) + x \cdot \frac{3}{8}\right) = x^2 + \frac{3}{8}x$$