UNIFORM CONVERGENCE

Contents

Suppose $f_n : \Omega \to \mathbb{R}$ or $f_n : \Omega \to \mathbb{C}$ is a sequence of real or complex functions, and $f_n \to f$ as $n \to \infty$ in some sense. Furthermore, suppose we know that f_n all have certain properties. What can we say about properties of f?

It turns out, that, at least for certain notions of convergence, we cannot really say anything about f. In order for limits of functions to preserve useful properties, like continuity or integrability, we need a certain type of convergence of functions known as *uniform convergence*.

1. Uniform Convergence

If $f_n : \Omega \to \mathbb{R}$ (everything works the same for $\mathbb{C}$) is a sequence of functions, we say that f_n converges to f pointwise on Ω if, for each $x \in \Omega$, $f_n(x) \to f(x)$. In other words, once we fix x, the sequence $f_n(x)$ has a limit, and we declare $f(x)$ to equal that limit. This is probably the most naive and obvious definition of what it means for a sequence of functions to converge to a limit. Unfortunately, this notion of convergence is weak, in the sense that the limiting procedure fails to preserve a variety of useful properties:

- Suppose all the f_n are continuous on Ω. If $f_n \to f$ pointwise, then f may not be continuous. For example, consider $f_n(x) = x^n, \Omega = [0, 1]$. Then f_n pointwise converges to the function $f(x)$ where $f(x) = 0, 0 \le x < 1, f(1) = 1$. Obviously this function is not continuous at $x = 1$. If the fact that this discontinuity is at an endpoint of a closed interval bothers you, it is easy to modify this example so the discontinuity occurs at an interior point (just symmetrically extend f_n to $[0, 2]$).
- The above example also makes it clear that if the f_n are differentiable, then its pointwise limit may not be differentiable.
- A homework problem provides an explicit example of a sequence of functions f_n defined on a closed interval I which pointwise converges to f on I, but for which

$$\lim_{n \to \infty} \int_I f_n \, dx \ne \int_I f \, dx.$$

  In other words, in general it is not possible to interchange a limit and an integral sign. Actually, it is not even clear that the integral on the right exists (namely, a pointwise limit of Riemann-integrable functions may not be Riemann-integrable).

Because pointwise convergence fails to preserve a variety of useful properties, we will define a stronger notion of convergence, and then prove that some of these properties are preserved under uniform convergence. However, certain properties still will not be preserved under uniform convergence; evidently limit processes can destroy lots of useful properties!

Suppose $f_n : \Omega \to \mathbb{R}$ is a sequence of functions and $f : \Omega \to \mathbb{R}$ is another function such that for all $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that for all $n > N, |f_n(x) - f(x)| < \varepsilon$ for all $x \in \Omega$. Then we say that f_n *converges uniformly* to f on Ω. When we write $N = N(\varepsilon)$, we emphasize the fact that while N is allowed to depend on ε, it **does not depend** on x.

What's the difference between this definition and pointwise convergence? We can rewrite the definition of pointwise convergence as saying that for each $x \in \Omega$ and $\varepsilon > 0$, there exists some $N = N(\varepsilon, x)$ such that for all $n > N, |f_n(x) - f(x)| < \varepsilon$. The difference is that in the uniform convergence definition, the N in that definition only depends on ε and not on x, while in the pointwise definition, N is allowed to depend on both ε and x.

Although this might seem like a minor difference, it actually makes a huge difference. In mathematics, uniformity usually refers to the property of a certain parameter not depending on another parameter – in this case, in the definition of uniform convergence, the parameter N does not depend on x. In many situations, uniformity is a good property to have, although its specific consequences vary from situation to situation.

The following more geometric interpretation of uniform convergence might be useful. The property that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in \Omega$ can be geometrically represented by saying that the graph of $f_n(x)$ is always within a band of width ε in either vertical direction of the function $f(x)$.

Examples.

- If f_n converges uniformly to f, then f_n also converges pointwise to f. This is an exercise in the homework. In particular, to test whether f_n converges uniformly to some function, it suffices to first check if this sequence converges pointwise to f, and then to check whether f is the uniform limit of f_n.
- Let us directly check that the sequence of functions $f_n(x) = x^n$ does not converge uniformly to its pointwise limit $f(x)$. (Recall that $f(x) = 0$ if $0 \le x < 1, f(1) = 1$.) Select any $\varepsilon < 1$, like $\varepsilon = 1/10$, say. We claim there is no N such that $|f_n(x) - f(x)| < 1/10$ for all $x \in [0, 1]$. Indeed, if there were, then $f_n(x) < 1/10$ for all $x \in [0, 1)$, since $f(x) = 0$ for $x \in [0, 1)$. But clearly this is false, since $f_n(x)$ is a continuous function on $[0, 1]$, and $f_n(1) = 1$, so that for x close to 1, $f_n(x) > .1$.

Are there any convenient criteria for determining whether a sequence of functions uniformly converges? There are a few, but we will primarily be interested in one criterion, which we will apply to power series inside its disc of convergence. We say that a series $\sum f_n$ uniformly converges to f if its partial sums uniformly converge to f.

Theorem 1 (Weierstrass M-test). *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions. Suppose there exist a sequence of non-negative real numbers M_n such that $|f_n(z)| \le M_n$*

for all $z \in \Omega$ and $\sum M_n$ converges. Then $\sum f_n$ uniformly converges on Ω to a function $f(z)$.

Proof. First we check that $\sum f_n(z)$ pointwise converges. Indeed, for a fixed z, we have $\sum |f_n(z)| \leq M_n < \infty$ converges, so that $\sum f_n(z)$ converges absolutely, and hence converges to some limit, which we call $f(z)$. Then $\sum f_n(z)$ pointwise converges to $f(z)$.

We now check that $\sum f_n(z)$ uniformly converges to $f(z)$. Let $\varepsilon > 0$ be arbitrary, and let $s_n(z) = \sum_{k=0}^{n} f_k(z)$ be the nth partial sum of $\sum f_n(z)$. We want to show that there exists some N such that for all $n > N$, $|f(z) - s_n(z)| < \varepsilon$ for any $z \in \Omega$. Notice that

$$|f(z) - s_n(z)| = \left| \sum_{k=n+1}^{\infty} f_k(z) \right| \leq \sum_{k=n+1}^{\infty} |f_k(z)| \leq \sum_{k=n+1}^{\infty} M_k.$$

Recall that $\sum_{k=1}^{\infty} M_k$ converges to some limit, so this means that for n sufficiently large, $\sum_{k=n+1}^{\infty} M_k < \varepsilon$. So select N to be a bound for this to be true. $\square$

Example. One of the most useful applications of the Weierstrass M-test is that it shows a power series uniformly converges in any closed disc contained in its disc of convergence. Suppose $\sum a_n z^n$ is a power series with radius of convergence R, and let Ω be any closed disc $|z| < R'$, where $R' < R$. Recall that in the proof of the theorem which shows the existence of the radius of convergence, we saw that $\sum a_n R'^n$ is bounded from above by a convergent geometric series $\sum r^n$. (The exact argument involved using properties of lim sup.) In particular, in the course of that proof we showed that $|a_n z^n| < cr^n$ whenever $|z| \leq R'$, where $r < 1$ is some real number (which does depend on R', but not on $|z|$) and c some real constant depending on R'. Therefore we can take $M_n = cr^n$ in the Weierstrass M-test.

2. Properties of uniform convergence

Now that we've defined what uniform convergence means, and saw a few examples of sequences/series of functions which did and did not uniformly converge, let's prove some of the basic properties of uniform convergence. First, we will show that the uniform limit of a sequence of continuous functions is still continuous (unlike the $f_n(x) = x^n$ example):

Proposition 1. *Suppose f_n converges uniformly to f on Ω. Then f is continuous on Ω.*

Proof. The method of proof is a simpler version of the proof that a power series can be differentiated term-by-term. We want to show that f is continuous on Ω, so this means given any $a \in \Omega$, we want to show that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that if $|x - a| < \delta$ (and $x \in \Omega$), then $|f(x) - f(a)| < \varepsilon$.

The way we will prove this is by writing $|f(x) - f(a)|$ in a slightly funny way, apply the triangle inequality, and then use the hypotheses to individually bound each of the remaining pieces. (If you look at the proof of the fact that power series can be differentiated term-by-term, the strategy is exactly the same.) More precisely, we write

$$\begin{aligned} |f(x) - f(a)| &= |(f(x) - f_n(x)) + (f_n(x) - f_n(a)) + (f_n(a) - f(a))| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|. \end{aligned}$$

First, because $f(x)$ is the uniform limit of $f_n(x)$, we know that there exists an N such that for all $n > N$, $|f(x) - f_n(x)| < \varepsilon/3$ for all $x \in \Omega$. But we also know that each f_n is continuous, so there exists some $\delta > 0$ such that for all $|x - a| < \delta$, $|f_n(x) - f_n(a)| < \varepsilon/3$. (It is possible that δ depends on n.)

So, in summary, given $\varepsilon > 0$ and $x \in \Omega$, select n large enough so that both $|f(x) - f_n(x)| < \varepsilon/3, |f(a) - f_n(a)| < \varepsilon/3$. This is possible because of uniform convergence. Next, select $\delta > 0$ such that $|x - a| < \delta$ implies $|f_n(x) - f_n(a)| < \varepsilon/3$. Then $|f(x) - f(a)| < \varepsilon$ if $|x - a| < \delta$, by adding up these three bounds on the inequality above.

$\square$

The second result we prove is that interchange of a uniform limit of functions with an integral over a finite interval (versus an improper integral, like an integral over $\mathbb{R}$) is allowed. A homework exercise asks you to generalize this to contour integrals.

Proposition 2. *Suppose $f_n : [a, b] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly to f on $[a, b]$. Then*

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

(Actually, the theorem is still true for f_n just Riemann-integrable, but we use this more restrictive version because the proof has fewer technical details in it, and the continuous version will be general enough for our purposes.)

Proof. We already have shown that f is continuous, hence integrable. Notice that

$$\left| \int_a^b f(x) - f_n(x)\,dx \right| \le \int_a^b |f(x) - f_n(x)|\,dx.$$

However, we also know that given $\varepsilon > 0$, there exists N such that $|f(x) - f_n(x)| < \varepsilon$ for $n > N$. Therefore, when $n > N$,

$$\left| \int_a^b f(x) - f_n(x)\,dx \right| \le \int_a^b |f(x) - f_n(x)|\,dx \le (b - a)\varepsilon.$$

Since $b - a$ is finite, this implies that $\lim_{n \to \infty} \int_a^b f(x) - f_n(x)\,dx = 0$, which is what we wanted to prove. $\square$

The contour integral version of this proposition is the primary reason why we are interested in uniform convergence: in one of the next major theorems we prove we will want to be able to justify switching a limit and integral operation.

As a matter of fact, you can check that on the question on last week's HW which required justifying limit and integral, that setup is just a special case of the contour integral version of this proposition: in other words, knowing this theorem would have saved a substantial amount of work! (On the other hand, if you did carefully

justify why switching limit and integral was legal, you basically did this proof for that particular example.)

Finally, one might ask whether uniform limits preserves differentiability, and if the limit of derivatives is the derivative of the limit. Unfortunately, in general the answer is no, as the following example shows:

Example. Let $f_n = \frac{1}{n} \sin nx$. Then f_n uniformly converges to $f(x) = 0$ on $x \in [0, 2\pi]$, say: indeed, notice that $|f_n(x) - 0| < 1/n$ for all $x \in \mathbb{R}$. However, $f_n'(x) = \cos(nx)$, which does not pointwise converge to any function, let alone converge to $f'(x) = 0$.

There is a slightly more restrictive case where uniform limits does preserve differentiability, however. We mention the theorem more out of completeness than anything else:

Proposition 3. *Suppose f_n uniformly converges to f on (a, b), and f_n' converges uniformly to g on (a, b). Then f is differentiable, and $f' = g$.*

The reason the last example does not contradict this proposition is because the derivatives of f_n did not uniformly converge (let alone pointwise converge) to any function.

Actually, another reason for mentioning this proposition is that we will see the situation with holomorphic functions is much simpler: a uniform limit of holomorphic functions is always holomorphic, and we can interchange the limit and differentiability operations. This is one example of how holomorphic functions are more well-behaved than their real differentiable counterparts.