INTEGRATION OF HOLOMORPHIC FUNCTIONS: CAUCHY'S THEOREM

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We will now consider the question of what happens when we integrate holomorphic functions. More specifically, we will prove a variety of theorems of the type 'If γ satisfies ... and f is holomorphic on γ and its interior, then $\int_\gamma f\,dz = 0$'. We will also prove related theorems telling us that in certain situations holomorphic functions have primitive functions on certain sets.

The starting point is a theorem of Goursat, which can be considered a precursor to the more general theorems we will prove. It is a special case of those general theorems, but because Goursat's Theorem is used as an ingredient in the proofs of those theorems it is not made obsolete by those theorems.

A surprising number of deep and non-trivial results follow from the theorems we will obtain. That the Fundamental Theorem of Algebra will be one of the easy consequences of these theorems shows that these theorems are very powerful.

1. Goursat's Theorem

Suppose γ is a path in $\mathbb{C}$ which is a triangle: that is, it is a simple closed path consisting of three (non-parallel) line segments. Suppose that f is holomorphic on an open set Ω which contains both γ and its interior. Then the following is true:

Theorem 1 (Goursat's Theorem)**.** *With the hypotheses as above,* $\int_\gamma f\,dz = 0$.

- There is a slight variation of this theorem, also known as Goursat's Theorem, which is often proven as an alternative in other sources. In that variation, the path γ is not a triangle, but a rectangle, often with sides parallel to the real and imaginary axis. We prove the triangle version because the proof is almost identical, and it is slightly easier to prove corollaries of the triangle version over the rectangle version. Nevertheless the triangle version follows fairly easily from the rectangle version.
- The hypothesis that f be holomorphic not only on the path of integration γ but also its interior is crucial. For example, if $f(z) = 1/z$, and γ is a triangle whose interior contains 0, then we will soon be able to prove that $\int_\gamma 1/z\,dz = 2\pi i \neq 0$.
- In some sense, there is a single-variable real version of this theorem, but it is trivial: if γ is a closed simple path in $\mathbb{R}$, then it must consist of only a single point, in which case a real analogue of this theorem would read $\int_a^a f(x)\,dx = 0$, which is obviously true.

On the other hand, if we think about a vector calculus line integral variation of this theorem, then we know that such a theorem is false. More precisely, if $\mathbf{F}$ is any differentiable vector field (say, C^1, or even C^∞), there is no guarantee that integrating it on a triangle or simple closed path will give 0. As a matter of fact we know that theorems of this type for vector fields will only be true if that vector field is conservative, which is not satisfied by most vector fields.

- One of the textbook exercises (which is assigned this week) shows that under the additional hypothesis of f being C^1 on γ and its interior (we do not yet know that if f is holomorphic, then f' is continuous), Goursat's Theorem can be proven using Green's Theorem from vector calculus. Part of the reason for the method of proof the text and we give is that we do not need to assume that f' is continuous.

Proof. The overall strategy of the proof is to estimate $|\int_\gamma f\,dz|$ by a sequence of inequalities whose upper bounds tend to 0. Without loss of generality, we assume that γ has positive orientation. We begin by splitting the triangle γ into four smaller, similar triangles, by connecting the midpoints of the three sides of γ. Let the boundaries of these triangles also have positive orientation, and call them $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$. Then

$$\int_{T_1^{(1)}} f(z)\,dz + \int_{T_2^{(1)}} f(z)\,dz + \int_{T_3^{(1)}} f(z)\,dz + \int_{T_4^{(1)}} f(z)\,dz = \int_\gamma f(z)\,dz,$$

because the integrals on all the interior edges cancel each other out. If we take absolute values of this equation and apply the triangle inequality, we get

$$\left|\int_\gamma f(z)\,dz\right| \le \sum_{i=1}^{4}\left|\int_{T_i^{(1)}} f(z)\,dz\right|.$$

Let $T^{(1)}$ be the one of the four smaller triangles for which $|\int_{T_i^{(1)}} f(z)\,dz|$ is maximal. Then we have

$$\left|\int_\gamma f(z)\,dz\right| \le 4\left|\int_{T^{(1)}} f(z)\,dz\right|.$$

This is the first of the inequalities we are looking for. In general, given a triangle $T^{(n)}$, we let $T^{(n+1)}$ be the smaller triangle obtained by the above procedure applied to $T^{(n)}$. If we let d_n, p_n be the diameter and perimeter of the triangle $T^{(n)}$, then by geometry, $d_n = 2^{-n}d, p_n = 2^{-n}p$, where d, p are the diameter and perimeter of the original triangle γ.

Repeatedly applying this construction, we get a sequence of triangles $T^{(n)}$ satisfying

$$\left|\int_{T^{(n)}} f(z)\,dz\right| \le 4\left|\int_{T^{(n+1)}} f(z)\,dz\right|,$$

which together imply

$$\left|\int_\gamma f(z)\,dz\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,dz\right|.$$

Also, the interior of $T^{(n)}$ together with its interior forms a closed, bounded set (compact set), say T_n, and these sets form a nested sequence of subsets $T_0 \supset T_1 \supset \ldots$. Since the diameters of these sets tends towards 0 (since $d_n = 2^{-n}d$), by an earlier theorem this implies that there exists a unique point z_0 contained in each of the sets T_n. Since z_0 is contained in T_0, which is the original triangle γ and its interior, this implies that f is holomorphic at z_0, since we assumed f is holomorphic at an open set containing γ and its interior.

Since f is holomorphic at z_0, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where $\psi(z) \to 0$ as $z \to z_0$. (Recall this was one of the alternate formulations for what differentiable/holomorphic means.) If we integrate $f(z)$ over $T^{(n)}$, using this expression for the integrand $f(z)$, we get

$$\int_{T^{(n)}} f(z)\ dz = \int_{T^{(n)}} f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)\, dz.$$

Notice that, as functions of z, both $f(z_0)$ (which is constant) and $f'(z_0)(z-z_0)$ (which is linear) have primitive functions on all of $\mathbb{C}$, hence also on Ω, and since $T^{(n)}$ is a closed curve, the integrals of these two functions are both 0. Therefore

$$\int_{T^{(n)}} f(z)\ dz = \int_{T^{(n)}} \psi(z)(z - z_0)\ dz.$$

If we wanted an exact value for this integral, this would not be particularly useful, since we do not know much about $\psi(z)$, but we only want an estimate. We estimate the integral on the right by using the ML-lemma. Notice that $|z - z_0| \leq d_n$ for all $z \in T^{(n)}$, because both z, z_0 are on the triangle $T^{(n)}$ or its interior. Let $\varepsilon_n = \sup_{z\in T^{(n)}} \psi(z)$ be the supremum of $\psi(z)$ on $T^{(n)}$. Since $\psi(z) \to 0$ as $z \to z_0$, and $T^{(n)} \to z_0$ as $n \to \infty$, this implies that $\varepsilon_n \to 0$ as $n \to \infty$. Finally, the length of $T^{(n)}$ is just p_n. Applying the ML-lemma with these estimates gives

$$\left| \int_{T^{(n)}} \psi(z)(z - z_0)\ dz \right| \leq d_n \varepsilon_n p_n = \varepsilon_n 4^{-n} dp.$$

Therefore

$$\left| \int_\gamma f(z)\, dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)\, dz \right| \leq \varepsilon_n.$$

Since $\varepsilon_n \to 0$ as $n \to \infty$, this implies that

$$\int_\gamma f(z)\, dz = 0,$$

as desired. $\square$

Corollary 1 (Rectangle version of Goursat's Theorem). *Let γ be a rectangle in $\mathbb{C}$, and let f be a function holomorphic on an open set containing γ and its interior. Then $\int_\gamma f(z)\, dz = 0$.*

Proof. Cut the rectangle into two triangles by drawing a single diagonal. Apply the triangle version of Goursat's Theorem to each triangle (which has orientation induced from the rectangle), add these two integrals, and notice that the integral along the diagonal cancels out. □

As a matter of fact, it is evident that this method will work with any polygon, since you can triangulate any polygon (this geometric fact is not entirely obvious, but we will only use it in situations where the triangulation is obvious) and apply the triangle version of Goursat's Theorem to each triangle in a triangulation.

2. Cauchy's Theorem: the disc version

We will now put Goursat's Theorem to use to prove a series of increasingly surprising results about holomorphic functions. The first theorem we prove probably does not seem too surprising, but it underlies several subsequent theorems. The method of proof also provides a good illustration of just why Goursat's Theorem is useful.

Theorem 2. *Suppose $f(z)$ is holomorphic in an open disc Ω. Then $f(z)$ has a primitive in Ω.*

Proof. To prove the theorem, we explicitly construct a primitive function $F(z)$ on Ω. We will need to prove that $F(z)$ is holomorphic and that $F'(z) = f(z)$ after defining $F(z)$.

The idea behind the definition of $F(z)$ is very similar to how one defines a potential function for a path-independent vector field. For ease of notation, suppose Ω is centered at 0 (if not, just translate everything to ensure 0 is the center of the disc). Let $z \in \Omega$. Let γ_z be the path joining 0 to z by first moving horizontally from 0 to $\operatorname{Re} z$ and then vertically from $\operatorname{Re} z$ to z. It is clear from the geometry of a disc that this path is always contained in Ω. We then define $F(z)$ as the contour integral of f along γ_z:

$$F(z) = \int_{\gamma_z} f(w)\, dw.$$

Because the way we constructed γ_z is unique, this definition is well-defined. We now want to prove that F is holomorphic and that $F'(z) = f(z)$; in other words, for any $z \in \Omega$, we want to show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

Let us examine $F(z+h) - F(z)$ more closely. If we draw the contours which appear in the definition of $F(z+h), F(z)$, we see that $F(z+h) - F(z)$ can be described as the integral of $f(w)$ along three sides of a narrow thin trapezoid: we start at z, move vertically to the real axis (and reach $\operatorname{Re} z$), move horizontally by $\operatorname{Re} h$, and then move vertically to $z + h$. (If $z, z + h$ have the same real part, we get a degenerate trapezoid, but it is easy to check that what we claim below still holds true.) If we complete this trapezoid by drawing the line segment from $z + h$ to z, we obtain a genuine trapezoid. Furthermore, we can split it into a rectangle and triangle, and by applying Goursat's Theorem to the rectangle and triangle, we find the integral of $f(z)$ over this completed trapezoid is 0. Therefore $F(z+h) - F(z)$ is equal to the integral of f along the line segment connecting z to $z+h$. (Notice that the geometry of the circle ensures that all the trapezoids, rectangles, triangles, and paths mentioned above lie within Ω, so that Goursat's Theorem is indeed applicable.)

Let η be the line segment connecting z to $z+h$. We have shown that

$$F(z+h) - F(z) = \int_\eta f(w)\,dw$$

via a clever application of Goursat's Theorem. Since f is holomorphic in Ω, it is also continuous in Ω, so we can write $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $w \to z$. Then

$$\int_\eta f(w)\,dw = \int_\eta f(z) + \psi(w)\,dw = f(z)\int_\eta 1\,dw + \int_\eta \psi(w)\,dw.$$

Notice that $\int_\eta 1\,dw = z+h-z = h$. On the other hand, we can estimate $\int_\eta \psi(w)\,dw$ using the ML-lemma:

$$\left|\int_\eta \psi(w)\,dw\right| \le |h| \sup_{z\in\eta} |\psi(w)|.$$

Notice that as $h \to 0$, $\sup_{z\in\eta} |\psi(w)| \to 0$, because $\psi(w) \to 0$ as $w \to z$. Therefore

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h\to 0} \frac{\int_\eta f(w\,dw}{h} = \lim_{h\to 0} \frac{f(z)h + |h|\sup_{z\in\eta}|\psi(w)|}{h} = f(z),$$

as desired. $\square$

Some straightforward and useful corollaries follow:

Corollary 2 (Cauchy's Theorem, disc version)**.** *Suppose f is holomorphic in a disc Ω. Let γ be any closed path in Ω; then $\int_\gamma f\,dz = 0$.*

Proof. By the previous theorem, f has a primitive in Ω. Then by an earlier theorem, $\int_\gamma f\,dz = 0$ for any γ closed curve contained in Ω. $\square$

Corollary 3 (Goursat's Theorem, circle version)**.** *Suppose f is holomorphic on an open set Ω containing a circle C and its interior. Then $\int_C f\,dz = 0$.*

Proof. Since Ω is open and C is compact, we can increase the radius of C slightly to get another circle C' whose interior is an open disc still entirely contained in Ω. (If you are not familiar with how to prove facts like this, try it as an exercise!) Apply Cauchy's Theorem to C which sits inside this open disc for which f is holomorphic to prove this corollary. $\square$

A natural question to ask after proving these theorems is whether or not the hypothesis that f be holomorphic in a disc is essential. What happens if instead we know that f is holomorphic in some other geometric set, such as the interior of a rectangle?

One can easily check that the entire proof of Theorem 2 generalizes to say, a rectangle whose sides are parallel to the coordinate axes. Given a holomorphic f in such a region, we can apply the same construction for F to prove a rectangle version of Theorem 2. It is a little harder to see that the same construction still essentially works for, say, sets Ω which are triangles or rectangles whose axes are not parallel to the coordinate axes of the plane. For rectangles, we can instead use paths which are parallel to the sides of the rectangle.

Suppose we try to apply the same construction to yet a more general region Ω, such as the interior of a polygon. The additional difficulty which appears is that it may be impossible to define $F(z)$ as the line integral along a path which is horizontal and then vertical, because such a path might leave Ω. This can be rectified by instead choosing a polygonal path which stays entirely within Ω, but then one must prove that the definition of F is independent of the choice of polygonal path. Proving this essentially boils down to using the rectangle version of Goursat's Theorem and very careful and tedious book-keeping of two polygonal paths in Ω which start and end at the same point.

A type of contour for which we will want to be able to apply Cauchy/Goursat's Theorem is what the book calls a 'keyhole' contour. Suppose C is a circle and z a point inside C, and let C' be a circle centered at z contained entirely in the interior of C. If we connect C and C' with a thin straight corridor, and then excise the arcs on C and C' which meet this corridor, we have a keyhole contour. It is a bit harder to see that we can still apply the proofs above to conclude that if f is holomorphic on an open set containing a keyhole contour γ and its interior, then $\int_\gamma f\,dz = 0$. In practice this is possible, but at the expense of fairly tedious calculations involving polygonal curves and checking that one can define F using a contour integral in a well-defined way.

The upshot is that you can either take it on faith that these calculations exist, carry out these calculations yourself, or wait until later where we will be able to prove that Cauchy's Theorem holds for a keyhole contour, among other types of contours. In any case, we will want to use the fact that Cauchy's Theorem is true for keyhole contours in our later proofs.

Another good question is just what types of regions Ω we cannot extend Cauchy's Theorem to. One region clearly is the interior of an annulus. Indeed, $f(z) = 1/z$ is holomorphic over any annulus which does not contain 0, but we saw that $\int_\gamma 1/z\,dz \neq 0$ if γ is a circle centered at 0 (and on homework, more generally if γ is a circle which contains 0.) So an important point is that CAUCHY'S THEOREM IS NOT VALID FOR AN ANNULUS.

What exactly differentiates an annulus from, say, a circular disc, a triangle, a rectangle, or a keyhole? That is, what is the difference between these shapes which permits us to prove Cauchy's Theorem for certain shapes but for which Cauchy's Theorem is false for others? We will see that the underlying topology of the shape is that key difference. We give a brief description of just what that difference is, and a glimpse of a theorem we (probably) will prove later in the class.

Suppose Ω is some open connected subset of $\mathbb{C}$. We say that Ω is *simply connected* if given any two paths γ_1, γ_2 (which we assume are parameterized by z_1, z_2 defined on $[0,1]$) entirely contained in Ω with same starting point and end point z_0, z_1, there exists a continual deformation of γ_1 to γ_2 which preserves endpoints and stays entirely in Ω. More precisely, there exists a continuous function $H : [0,1] \times [0,1] \to \Omega$ such that $H(t,0) = z_1(t), H(t,1) = z_2(t), H(0,s) = z_0, H(1,s) = z_1$ for all $0 \leq t, s, \leq 1$.

You should think of $H(t, s)$ as a function which, for fixed s, represents a deformation of $H(t, 0) = z_1(t)$ to $H(t, 1) = z_2(t)$.

An alternate formulation of simply connected spaces is that a space is simply connected if any closed path in Ω can be continually deformed (in the sense above) to a point in Ω without leaving Ω. Intuitively, a simply connected space is a space which has no 'holes' in it.

Some of the following examples are not rigorous (in the sense that we will not yet rigorously prove what we claim), but the intuition should be clear.

Examples.

- Open discs are simply connected. We can prove this: given any closed path γ in an open disc, we can continually deform it to a point by picking an arbitrary point on the curve, and then linearly contracting the rest of the points towards that point.
- More generally it is clear that $\mathbb{R}^n$ is simply connected, as is any convex subset of $\mathbb{R}^n$, since we can apply the above idea to both these spaces.
- Keyhole contours are simply connected. This is a bit harder to prove, but intuitively is fairly clear. (For example, to deform a curve to a point, first linearly deform the curve to a circular arc running the length of the keyhole, and then deform along that circular arc. This idea can be formalized via the notion of homotopy equivalence.)
- An annulus is not simply connected. This is intuitively clear since the inner circle in an annulus encloses a hole which is not in that annulus. Later, we will prove this by proving that Cauchy's Theorem holds for simply connected sets, and we know that Cauchy's Theorem does not hold for an annulus.

What should you take away from this discussion? Here are some key points:

- Cauchy's Theorem is rigorously proven for open discs via what we did earlier. Without much difficulty one can adopt the argument to triangles, rectangles, as well as sectors of circles.
- With considerably more effort one could adopt the argument, right now, to keyhole contours, without appealing to non-trivial ideas from topology. This is fairly tedious, so we will assume Cauchy's Theorem holds for keyhole contours, and you can either try to work out a rigorous argument yourself, or wait until we prove Cauchy's Theorem for simply connected regions.
- As a practical matter, at this point you may freely cite Cauchy's Theorem for circles, triangles, rectangles, and keyhole contours. For other simply connected shapes, you should probably not cite Cauchy's Theorem, since we have not yet proven it for general simply connected shapes yet. In practice you will probably only need Cauchy's Theorem for circles, sectors of circles, triangles, rectangles, keyholes, and perhaps other very similar shapes in which a construction for a primitive is possible along the lines given above.
- Eventually we will prove Cauchy's Theorem holds for any simply connected open set Ω. Again, do not cite this theorem since we have not yet proven it. (As a practical note, this does not obselete the proof given above, because we will use the disc version of Cauchy's Theorem in a crucial way to prove the more general version.)

- Cauchy's Theorem IS NOT VALID for an annulus.
- There is a non-trivial and somewhat subtle connection between complex analysis (contour integration) and topology.