DIFFERENTIABILITY OF COMPLEX FUNCTIONS

Contents

1. Limit definition of a derivative

Since we want to do calculus on functions of a complex variable, we will begin by defining a derivative by mimicking the definition for real functions. Namely, if $f : \Omega \to \mathbb{C}$ is a complex function and $z \in \Omega$ an interior point of f, we define the *derivative* of f at z to be the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

if this limit exists. If the derivative of f exists at z, we denote its value by $f'(z)$, and we say f is*holomorphic* at z. If f is holomorphic at every point of an open set U we say that f is holomorphic on U.

This definition naturally leads to several basic remarks. First, the definition formally looks identical to the limit definition of a derivative of a function of a real variable, which is inspired by trying to approximate a tangent line using secant lines. However, in the limit as $h \to 0$, we are allowing h to vary over all complex numbers that approach 0, not just real numbers. One of the main principles of this class is that this seemingly minor difference actually makes a gigantic difference in the behavior of holomorphic functions.

We insist that z be an interior point of Ω to ensure that as we let $h \to 0$, we can approach h in any direction. This is similar to the fact that derivatives (at least, derivatives which are not one-sided) of real functions are only defined at interior points of intervals, not at the endpoints of closed intervals.

Why is the term holomorphic used instead of differentiable? We could use differentiable, or perhaps the more specific term complex differentiable, but the convention in mathematics is that the term holomorphic refers to differentiability of complex functions, not real functions. In particular, we will see that a complex function being holomorphic is substantially more restrictive than the corresponding real function it induces being differentiable.

Perhaps you might be wondering what exactly a limit of a complex function is. After all, when we say that $h \to 0$ as h ranges over complex numbers, what exactly do we mean? The rigorous definition is just the natural extension of the ε–δ definition used for real functions. More precisely, we say that $\lim\limits_{z \to z_0} f(z) = w$ if for all $\varepsilon > 0$

there exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - w| < \varepsilon$. In other words, if we specify any error ε, no matter how small it is, all values of $f(z)$ for z sufficiently close to z_0 will be within that error of w.

The fact that the formal definition of derivatives of complex functions is so similar to real functions turns out to be handy for proving many of the basic properties of the complex derivative. In particular, many of the proofs of these properties in the real case transfer over, in almost identical fashion, to the complex case. For example, one can show the following:

Suppose $f, g : \Omega \to \mathbb{C}$ are differentiable at z. Then

- $f + g$ is differentiable at z, and $(f + g)'(z) = f'(z) + g'(z)$,
- if c is any complex number, then cf is differentiable at z, and $(cf)'(z) = cf'(z)$,
- (fg) is differentiable at z, and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ (product rule),
- if f is differentiable at z and g is differentiable at $f(z)$, then $(g \circ f)$ is differentiable at z, and $(g \circ f)'(z) = g'(f(z))f'(z)$. (chain rule)

So, the good news is that all of the standard differentiation rules carry over to the complex case with no formal change. Let's use these properties and the limit definition of a derivative to calculate some derivatives.

Examples.

- Let $f(z) = z$. Then
$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{z + h - z}{h} = 1.$$
Therefore $f(z) = z$ is differentiable on all of $\mathbb{C}$ and has derivative $f'(z) = 1$. This is exactly identical to the real case.
- More generally, suppose n is a positive integer. We claim that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$. We can prove this either using the binomial theorem to expand $(z + h)^n$, or by induction. Let's do the latter.

  Recall that to prove a statement parameterized by positive integers using induction, we prove the base case (usually $n = 1$), and then prove that if the statement is true for n it is also true for $n + 1$. We already proved the $n = 1$ case in the prior example. Suppose we know that if $f(z) = z^n$, then $f'(z) = nz^{n-1}$. Using the product rule (which we assume has been proven for us), if $f(z) = z^{n+1}$, then $f'(z) = (z^n)'z + z^n(z)' = nz^{n-1} \cdot z + z^n = (n + 1)z^n$, as desired. Furthermore, $f(z) = z^n$ is differentiable on all of $\mathbb{C}$.
- From the above calculations and the basic properties we mentioned earlier, differentiating a polynomial with complex coefficients is formally identical to differentiation of real polynomials. For example, if $f(z) = iz^4 + (2-3i)z^3 + 7z$, then $f'(z) = 4iz^3 + (6 - 9i)z^2 + 7$.
- Similarly, one can show that if n is a negative integer, then $f(z) = z^n$ has derivative $f'(z) = nz^{n-1}$, although of course this time $f(z)$ is differentiable on $\mathbb{C} - 0$, since z^n is undefined at $z = 0$ if $n < 0$.
- Even though we have proven the 'power rule' for integer exponents on z, notice that we have not said anything about whether this rule is true for non-integer exponents. As a matter of fact, notice that we do not yet have any good definition of what non-integer exponents mean!

- We will postpone the calculation of derivatives of e^z and related functions until we study power series.

2. Holomorphic functions, the Cauchy-Riemann equations

Let $f : \Omega \to \mathbb{C}$ be any complex function. Since we can identify $\mathbb{C}$ with $\mathbb{R}^2$, any such function automatically induces a related function $f : \Omega \to \mathbb{R}^2$, where now we think of Ω as being a subset of $\mathbb{R}^2$ instead of $\mathbb{C}$. If f is holomorphic, does this imply anything about real differentiability of its associated real function? As a matter of fact, yes, it does, and it turns out that this connection is not merely a curiosity; many deep theorems about certain real functions can be proven by appealing to this connection!

Examples.

- Suppose $f(z) = z$. Then we may write $f(x + yi) = x + yi$, so f induces the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (x, y)$.
- Suppose $f(z) = z^2$. Then we may write $f(x+yi) = (x+yi)^2 = (x^2-y^2)+2xy\cdot i$, so f induces the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (x^2 - y^2, 2xy)$.
- Suppose $f(z) = e^z$. Assuming basic properties of complex exponentials (which we will prove soon), $e^{x+yi} = e^x \cdot e^{yi} = e^x(\cos y + i \sin y)$. Therefore $f(z) = e^z$ induces the function $f(x, y) = (e^x \cos y, e^x \sin y)$.
- Suppose $f(z) = 1/z$. Then we may write $f(x + yi) = \dfrac{1}{x + yi} = \dfrac{x}{x^2 + y^2} - \dfrac{y}{x^2 + y^2}i$, so f induces $f(x, y) = \left(\dfrac{x}{x^2 + y^2}, -\dfrac{y}{x^2 + y^2}\right)$, and this real function is defined on $\mathbb{R}^2 - (0, 0)$.
- Suppose $f(z) = |z|$. Then we may write $f(x + yi) = \sqrt{x^2 + y^2}$, so f induces $f(x, y) = (\sqrt{x^2 + y^2}, 0)$.
- Suppose $f(z) = \operatorname{Im} z$. Then we may write $f(x + yi) = y$, so f induces $f(x, y) = (y, 0)$.

In general, if $f(x + yi) = u(x, y) + v(x, y)i$, where u, v are functions from some subset of $\mathbb{R}^2$ to $\mathbb{R}$, then f induces a real function from some subset of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $f(x, y) = (u(x, y), v(x, y))$.

Suppose f is holomorphic at z, and write $f(x + yi) = u(x, y) + v(x, y)i$. Then we can compute the derivative of f at z using the limit definition by either letting h approach 0 along the real axis or along the imaginary axis. If we approach along the real axis,

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h\to 0} \frac{f(x + h + yi) - f(x + yi)}{h} \\ &= \lim_{h\to 0} \frac{u(x + h, y) + v(x + h, y)i - (u(x, y) + v(x, y)i)}{h}. \end{aligned}$$

Separating the last expression into real and imaginary parts,

$$f'(z) = \lim_{h\to 0} \frac{u(x + h, y) - u(x, y)}{h} + \frac{v(x + h, y) - v(x, y)}{h} i.$$

These two expressions appear in multivariable calculus: they are the partial derivatives of $u(x, y), v(x, y)$ with respect to x. So we have shown that if f is holomorphic at z, then u_x, v_x exist at (x, y), and we have an equation which relates $f'(z)$ to u_x, v_x: namely,

$$f'(z) = u_x(x, y) + v_x(x, y)i.$$

If we repeat the same procedure except this time we let $h \to 0$ along the imaginary axis, then letting $h' \to 0$ along reals, we get

$$\begin{aligned}
f'(z) &= \lim_{h' \to 0} \frac{f(z + ih') - f(z)}{ih'} = \frac{1}{i} \lim_{h' \to 0} \frac{f(x + (y + h')i) - f(x + yi)}{h} \\
&= \frac{1}{i} \lim_{h' \to 0} \frac{u(x, y + h') + v(x, y + h')i - (u(x, y) + v(x, y)i)}{h'} \\
&= \frac{1}{i} \lim_{h' \to 0} \frac{u(x, y + h') - u(x, y)}{h'} + \frac{v(x, y + h') - v(x, y)}{h'} i \\
&= v_y(x, y) - u_y(x, y)i.
\end{aligned}$$

Therefore we have two alternate expressions for $f'(z)$, and since they must be equal, and u_x, u_y, v_x, v_y are all real numbers, we have proven the following theorem:

Theorem 1. *Suppose $f(x + yi) = u(x, y) + v(x, y)i$ is holomorphic at $z = x + iy$. Then the partial derivatives u_x, u_y, v_x, v_y all exist at (x, y), and at (x, y) they satisfy the pair of partial differential equations*

$$u_x = v_y, u_y = -v_x.$$

More generally, if f is holomorphic on an open set Ω, then u_x, u_y, v_x, v_y exist on Ω, and $u_x = v_y, u_y = -v_x$ holds true on all of Ω. We call this pair of partial differential equations the Cauchy-Riemann equations.

At this point, we know that any holomorphic function must have real and imaginary parts which satisfy the Cauchy-Riemann equations. One use of this fact/theorem is that it lets us show that certain functions are not holomorphic:

Examples.

- Consider the complex conjugation function $f(z) = \overline{z}$. Then it induces the functions $u(x, y) = x, v(x, y) = -y$. Computing partial derivatives, we find that $u_x = 1, v_y = -1, u_y = 0, v_x = 0$, so the CR equation $u_x = v_y$ is never satisfied for any z. Therefore $f(z)$ is not holomorphic at any point of $\mathbb{C}$. Notice that this might seem somewhat surprising, since the real and imaginary parts of $f(z) = \overline{z}$ look so simple and well-behaved. Indeed, the CR equations are highly nontrivial and will only be satisfied for very special pairs of u, v.
- Let $f(z) = \operatorname{Im} z$. Then f induces the functions $u(x, y) = y, v(x, y) = 0$. Since $u_y = 1, v_x = 0$, the CR equations are never satisfied, since $u_y = -v_x$ is impossible at any point z. Therefore $f(z) = \operatorname{Im} z$ is never holomorphic at any point of z.

These examples illustrate that even if you construct a complex function $f(x+yi) = u(x, y)+v(x, y)i$ using ‘well-behaved’ $u(x, y), v(x, y)$, f may still not be holomorphic. As a matter of fact, notice that even in perhaps the best possible situation, where

u, v are polynomials, as in the above examples, f does not have to be holomorphic. That the CR equations are so special turn out (in the end) to be the source of all the strong properties that holomorphic functions satisfy.

A natural question is whether the converse of the above theorem holds: namely, if $u(x,y), v(x,y)$ satisfy the CR equations, if $f(z) = u(x,y) + v(x,y)i$ is holomorphic. It turns out that under certain fairly weak conditions, the converse is indeed true, so that the CR equations essentially completely determine whether a function f is holomorphic. However, before giving the statement and proof we need to review a few concepts from the calculus of real functions.

3. Differentiability of real functions

We spend a lot of time in single-variable calculus computing derivatives, and thinking about the geometric significance of derivatives. In multivariable calculus, we discuss attempts to generalize derivatives, like partial derivatives and directional derivatives. However, many introductory multivariable calculus classes do not discuss what the derivative of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is, or if they do, they do not discuss it in depth. We want to quickly review what the derivative of such a function is.

We begin by interpreting derivatives of single variable functions in a slightly different way than what a calculus student is accustomed to. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x. Then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Another way of saying this is that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - hf'(x)}{h} = 0.$$

In other words, if we define a new function $r(h)$ such that $f(x+h) - f(x) - hf'(x) = hr(h)$, then $\lim_{h\to 0} r(h) = 0$. So we can redefine what it means for $f(x)$ to be differentiable at x as follows: f is differentiable at x if and only if there exists a real number $f'(x)$ such that we can write

$$f(x+h) = f(x) + hf'(x) + hr(h),$$

where $r(h) \to 0$ as $h \to 0$. Sometimes, instead of $hr(h)$, we will use $|h|r(h)$; evidently this does not change whether a function is differentiable or not.

Examples.

- Consider $f(x) = x^2$. We know that $f'(x) = 2x$; let us see how this new interpretation of a derivative works in this example. Let $x = 3$. Then $f'(3) = 6$, and if we write

  $$f(3+h) = f(3) + hf'(3) + hr(h),$$

  we can solve for $r(h)$: $f(3+h) = (3+h)^2 = h^2+6h+9$, and $f(3) = 9, f'(3) = 6$, so $h^2 + 6h + 9 = 9 + 6h + hr(h)$, so $r(h) = h$. Notice that $r(h) = h \to 0$ as $h \to 0$, as expected.

- Consider $f(x) = \sin x$, and let $x = 0$. Then $f(0) = 0, f'(0) = 1$. We can write

$$f(0+h) = f(0) + hf'(0) + hr(h),$$

which becomes $\sin h = 0 + h + hr(h)$. Therefore $r(h) = \frac{\sin h}{h} - 1$. Notice that $\lim_{h\to 0} r(h) = 0$, as expected.

Geometrically, one should think of this new interpretation of derivative as saying that a function f is differentiable at x if and only if there is a good linear approximation to f at x, in the sense that this linear approximation has error which is of lower order than the linear approximation itself.

How does this help with defining a derivative for functions of several real variables? Notice that we cannot imitate the limit definition in a naive way, since we cannot divide by ordered tuples of real numbers. So instead we mimic this description of a derivative, replacing the constant $f'(x)$ by a linear function of several real variables, which is known as a *linear transformation* in linear algebra.

A linear transformation is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ which has the form

$$T(x_1, x_2, \ldots, x_m) = (a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m, a_{21}x_1 + \ldots + a_{2m}x_m, \ldots),$$

where the a_{ij} are real numbers. We typically represent T using an $n \times m$ matrix with a_{ij} as its entries. For example, the function $T(x, y) = (2x + 4y, -x + 3.5y, 7x)$ is represented by the matrix

$$\begin{bmatrix} 2 & 4 \\ -1 & 3.5 \\ 7 & 0 \end{bmatrix}.$$

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be any real function. (More generally, f only needs to be defined on a subset of $\mathbb{R}^m$). Then we say that f is differentiable at a point $\mathbf{x} = (x_1, \ldots, x_m)$ if there exists a linear transformation $T_{\mathbf{x}} = T : \mathbb{R}^m \to \mathbb{R}^n$ such that if we write

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + T(\mathbf{h}) + |\mathbf{h}|r(\mathbf{h}),$$

then $\lim_{\mathbf{h}\to\mathbf{0}} r(\mathbf{h}) = \mathbf{0}$, where this limit takes place in $\mathbb{R}^m$. We call the linear transformation $T_x = T$ the *derivative* (sometimes total derivative) of f at $\mathbf{x}$.

This definition is difficult to work with directly. Fortunately, in most situations it is fairly easy to compute, because it turns out that real differentiability is closely related to the existence of partial derivatives. In particular, we have the following theorem, which we will not prove, from real analysis:

Theorem 2. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be any function. Write $f = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)$. Let $\mathbf{x} = (x_1, \ldots, x_m)$ be a point of $\mathbb{R}^m$. If the partial derivatives $\partial_j f_i$ exist for all i, j in some open ball containing $\mathbf{x}$ and are also all continuous at $\mathbf{x}$, then f is differentiable at $\mathbf{x}$.*

Furthermore, if f is differentiable at $\mathbf{x}$, then all the partial derivatives $\partial_j f_i$ exist, and the derivative of f at $\mathbf{x}$ is given by the Jacobian matrix

$$J(x_1, \ldots, x_m) = (\partial_j f_i(x_1, \ldots, x_m))_{ij}.$$

The Jacobian matrix is just the matrix you get by computing the gradient of each of the component functions of f and then evaluating them at the point $(x_1, \ldots, x_m)$, and taking these gradients as the rows of the Jacobian matrix.

The hypothesis that the partial derivatives must exist in a neighborhood of $\mathbf{x}$ is essential. It is not enough to know that the partial derivatives exist at $\mathbf{x}$; there are examples of functions which have partial derivatives at a point $\mathbf{x}$ but are not differentiable there.

The property of a function having continuous partial derivatives appears so frequently that we give it a name. If Ω is an open set in $\mathbb{R}^n$, we say that a function $f : \mathbb{R}^m \to \mathbb{R}$ is in $C^k(\Omega)$ if all the partial derivatives of order k exist and are continuous at every point of Ω. A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is in $C^k(\Omega)$ if each of its component functions are $C^k(\Omega)$. A function which is in $C^k(\Omega)$ for every $k \geq 1$ is in $C^\infty(\Omega)$.

The theorem above says, for example, that any C^1 function on an open set Ω is differentiable on Ω, with derivative given by the Jacobian matrix.

- Let $f(x, y) = (x^2 + xy, -2xy)$. Since the component functions are C^∞ (being polynomials), f is real differentiable everywhere, and its derivative is given by
$$J(x, y) = \begin{bmatrix} 2x + y & x \\ -2y & -2x \end{bmatrix}.$$
- Let $f(x, y) = (e^x \cos y, e^x \sin y)$. (This is the real function that $f(z) = e^z$ induces.) Then f is real differentiable everywhere, since its component functions are C^∞, and its derivative is given by
$$J(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$
- Suppose $f(x, y) = (u(x, y), v(x, y))$, and u, v are C^1 functions that satisfy the CR equations on an open set Ω. Then f is differentiable at Ω, and has derivative given by
$$J(x, y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$
However, the CR equations tell us that $u_x = v_y, u_y = -v_x$. In other words, the CR equations are equivalent to saying that the Jacobian matrix of $f(x, y) = (u(x, y), v(x, y))$ is an anti-symmetric matrix. For instance, in the previous example, the Jacobian matrix is anti-symmetric, so $f(z) = e^z$ satisfies the CR equations.
- Let $f(x)$ be a function of a single variable. Then the Jacobian matrix is just the 1×1 matrix $[f'(x)]$, so the Jacobian matrix can be thought of as a generalization of the ordinary notion of derivative for a single-variable real function.

4. A sufficient condition for holomorphy

With this background on real differentiability in hand, we can now prove a sufficient condition involving the CR equations for a complex function f to be holomorphic.

Before beginning, we remark that the description of a differentiable function as a function which has a good linear approximation (in the sense that $f(x+h) = f(x) + hf'(x) + |h|r(h)$ satisfies $r(h) \to 0$ as $h \to 0$) extends to holomorphic functions, where we now interpret $h \to 0$ as a limit in complex numbers.

Theorem 3 (Theorem 2.4 of Ch.1 in the text)**.** *Suppose $f(x+yi) = u(x,y)+v(x,y)i$ is a complex function defined on an open set Ω in $\mathbb{C}$. Suppose u, v are C^1 on Ω and satisfy the Cauchy-Riemann equations. Then f is holomorphic on Ω.*

Proof. We begin by remarking that if, say, $u(x,y)$ is C^1 on Ω, then u is differentiable on Ω. Therefore, its matrix is given by the Jacobian matrix $[u_x\, u_y]$, which is just the gradient, and the definition of differentiability can be re-expressed by saying that if we write

$$u(x+h_1, y+h_2) = u(x,y) + u_x(x,y)h_1 + u_y(x,y)h_2 + |h|r(h),$$

then $r(h) \to 0$ as $h = (h_1, h_2) \to (0,0)$. (In other words, the tangent plane to $u(x,y)$ is a good linear approximation to u near (x,y).)

Applying this to both u, v, we find that we can write

$$\begin{aligned} u(x+h_1, y+h_2) &= u(x,y) + u_x(x,y)h_1 + u_y(x,y)h_2 + |h|r_1(h) \\ v(x+h_1, y+h_2) &= v(x,y) + v_x(x,y)h_1 + v_y(x,y)h_2 + |h|r_2(h) \end{aligned}$$

where $r_1(h), r_2(h) \to 0$ as $h \to (0,0)$.

Since $f(z) = u + vi$, if we let $h = h_1 + ih_2$, we can write

$$f(z+h) - f(z) = u(x+h_1, y+h_2) - u(x,y) + i(v(x+h_1, y+h_2) - v(x,y)).$$

Plug in the two expressions above for $u(x+h_1, y+h_2) - u(x,y)$ and the analogous expression for v:

$$f(z+h)-f(z) = u_x(x,y)h_1+u_y(x,y)h_2+|h|r_1(h)+i(v_x(x,y)h_1+v_y(x,y)h_2+|h|r_2(h)).$$

The $|h|r_1(h)$ and $i|h|r_2(h)$ sum to a term $|h|r(h)$, where $r(h) \to 0$ as $h \to 0$, since $r_1(h), r_2(h) \to 0$ as $h \to 0$. For the remaining four terms, apply CR to convert the v_x, v_y terms to u_x, u_y terms:

$$u_x(x,y)h_1 + u_y(x,y)h_2 + i(v_x(x,y)h_1 + v_y(x,y)h_2) = u_xh_1 + u_yh_2 - iu_yh_1 + iu_xh_2.$$

However, notice we can factor the right hand side as $(u_x - iu_y)(h_1 + ih_2)$. Therefore, we can write

$$f(z+h) - f(z) = (u_x - iu_y)h + |h|r(h),$$

where $r(h)$ is a complex function satisfying $r(h) \to 0$ as $h \to 0$. Therefore f is differentiable, with derivative $u_x - iu_y = u_x + iv_x$, which is the expected value of the derivative. $\square$

A natural although somewhat subtle question is whether weaker conditions on u, v (weaker than being C^1) are sufficient to prove holomorphy. As a matter of fact, there are theorems with weaker hypotheses, but they are surprisingly non-trivial to prove. See this article by Grey and Morris in the American Mathematical Monthly for more information, if interested.

We conclude by proving a few more basic facts that use the CR equations. First, we show that if f is holomorphic in a region Ω, then considered as a real function it is also real differentiable there (as expected).

Theorem 4 (Proposition 2.3 in Chapter 1 of the text)**.** *Suppose f is a complex function holomorphic on a region Ω. Then considered as a real function from Ω to $\mathbb{R}^2$, f is real differentiable, and the determinant of the Jacobian matrix at a point (x, y) is equal to $|f'(x+yi)|^2$.*

Proof. To show that f is differentiable as a real function, we want to show that if (x, y) is a point in Ω, there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x+h_1, y+h_2) - f(x, y) = T(h_1, h_2) + |h|r(h),$$

where $|r(h)| \to 0$ as $h \to (0, 0)$. Since we know that f is holomorphic, we can write

$$f(z+h) - f(z) = hf'(z) + |h|r(h),$$

where $r(h) \to 0$ as $h \to 0$ in complex numbers. Thinking of this equation as relating real functions, we obtain

$$f(x+h_1, y+h_2) - f(x, y) = (h_1 + ih_2)(u_x + iv_x) + |h|r(h).$$

The first term on the right hand side is equal to $(h_1u_x - h_2v_x) + i(u_xh_2 + v_xh_1)$. Therefore T is given by the matrix

$$\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

(We used the CR equations in the above equality.) So f is real differentiable with Jacobian matrix above, and the determinant of this matrix is $u_x^2 + v_x^2 = |u_x + iv_x|^2 = |f'(z)|^2$, as desired. $\square$

The next proposition shows that holomorphic functions are very closely related to a special type of function called a *harmonic function*, which appear frequently in physics and engineering. We begin by defining what a harmonic function is:

Definition 1. *The differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ which operates on functions $f : \mathbb{R}^2 \to \mathbb{R}$ (or more generally, functions $f : \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^2$) is called the* Laplacian *in $\mathbb{R}^2$, and is sometimes written as Δ.*

Definition 2. *Let $f : \Omega \to \mathbb{R}$ be a real-valued function defined on an open set Ω in $\mathbb{R}^2$. If f is C^2 and $\Delta f = 0$, then we call f a* harmonic *function.*

Examples.

- Constant functions are harmonic, since their second order partial derivatives are always equal to 0.

- Any linear function, ie, function of the form $f(x,y) = ax + by + c$, is also harmonic.
- $f(x,y) = e^x \cos y$ is harmonic on $\mathbb{R}^2$. Indeed, $f_{xx} = e^x \cos y$, while $f_{yy} = -e^x \cos y$, so $f_{xx} + f_{yy} = 0$ is true everywhere.
- There are actually much more complicated functions which are harmonic, but we will not discuss them further, since they constitute a large portion of the study of partial differential equations. However, the next proposition shows that any holomorphic function produces harmonic functions automatically.

Proposition 1. *Suppose $f(z) = u + vi$ is holomorphic on an open set Ω, and suppose u, v are both C^2 (we will soon be able to remove this condition). Then u and v are both harmonic on Ω.*

Proof. Since f is holomorphic, the CR equations tell us that $u_x = v_y, u_y = -v_x$. Differentiate the first equation with respect to x and the second with respect to y to get $u_{xx} = v_{yx}, u_{yy} = -v_{xy}$. Since u, v are C^2, $v_{yx} = v_{xy}$, so $u_{xx} + u_{yy} = 0$; ie, u is harmonic on Ω. Something similar works to show that v is also harmonic on Ω. $\square$

The CR equations provide a non-obvious connection between harmonic functions (which on the surface seem to have nothing to do with complex numbers or holomorphic functions), which appear everywhere in physics and engineering, and holomorphic functions. As a consequence, several of the non-obvious theorems we prove for holomorphic functions during this class will have natural counterparts for harmonic functions. We will point some of these connections out, but one could fill many classes discussing harmonic functions and their relatives.