Math 43 Spring 2007

Functions of a Complex Variable

Midterm Exam

Monday, April 23, 2007

Your name (please print): ______________________________________

Instructions: This is an open book, open notes take home exam. You can use any printed books you like. **Use of calculators and internet is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise in a given problem.

The exam is to be submitted on Friday April 27 during the regular Math 43 class time. The exam consists of 11 problems. Your total exam score is the sum of your scores for the 10 problems best solved. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. ________ /10

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /10

6. ________ /10

7. ________ /10

8. ________ /10

9. ________ /10

10. ________ /10

11. ________ /10

Total: ________ /100

(1) Find $(\frac{\sqrt{3}}{2} + \frac{1}{2}i)^{2007}$. The answer should be given in the form $a + bi$ where $a, b \in \mathbb{R}$.

(2) Find the 6 roots of the equation $z^6 + 1 = 0$ and use them to factor $z^6 + 1$ into a product of 3 quadratic factors with real coefficients.

(3) Let $S \subset \mathbb{C}$ be a nonempty open set. Prove that every boundary point of S is an accumulation point of S.

(4) Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $\lim_{z \to z_0} f(z) = 0$. Using the ϵ, δ definition of the limit to prove that $\lim_{z \to z_0} (f(z))^2 = 0$. You are not allowed to use any theorems about the properties of the limits.

(5) Let

$$f(z) = \begin{cases} i\dfrac{z^2}{\overline{z}} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

Does f satisfy Cauchy-Riemann equations at zero? Prove your answer.

(6) Find all the values of $\cos^{-1}(2i)$.

(7) Find the maximal open subset S of $\mathbb{C}$ such that the function P. V. $(iz-1)^{3+i}$ is analytic in S.

(8) Find an entire function $f(z)$ such that $Ref(x+iy) = 3x^2 - 3y^2 - 2xy$ for all $x, y \in \mathbb{R}$.

(9) Let $f(z)$ be an entire function such that $f(0+iy) = \cos y + i \sin y$ for all $y \in \mathbb{R}$. Find $f(1+i)$. Prove your answer.

(10) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)|^2 = f(z)f(\overline{z})$ for all $z = x + i0, x \in \mathbb{R}$. Prove that $|f(z)|^2 = f(z)f(\overline{z})$ for all $z \in \mathbb{C}$.

(11) Prove that

$$v(x,y) = \operatorname{Im} \overline{exp\Big(\frac{1}{z^{2007}-1}\Big)}$$

is harmonic in the domain $\{(x,y)|x^2+y^2>1\}$.