Math 43 Spring 2007

Functions of a Complex Variable

Final Exam

Friday, June 1, 2007

Your name (please print): ____

Instructions: This is an open book, open notes take home exam. You can use any printed books you like. **Use of calculators and internet is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise in a given problem.

The exam is to be submitted on **Tuesday June 5 by 10 PM.** The best way to submit the exam is in person. I will be at the department during the regular work hours on Tuesday. I will also try to be in the office the last couple of hours before 10 PM on Tuesday. If you submit the exam when my office 304 Kemeny is locked, then please slide it under my office door and write the time you have finished working on the exam.

The exam consists of 11 problems. Your total exam score is the sum of your scores for the 10 problems best solved. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. ________ /15

2. ________ /15

3. ________ /15

4. ________ /15

5. ________ /15

6. ________ /15

7. ________ /15

8. ________ /15

9. ________ /15

10. ________ /15

11. ________ /15

Total: ________ /150

(1) For $a \in \mathbb{C}$ put $f_a : \mathbb{C} \to \mathbb{C}$ to be the function defined as follows

$$f_a(z) = \begin{cases} \frac{\sin z - z}{z^2} & \text{for } z \neq 0; \\ a & \text{for } z = 0. \end{cases}$$

Find all a such that the function f_a is entire. Prove your answer.

(2) Is 0 an isolated singular point of the function $f(z) = \dfrac{1}{\sinh^2(\frac{1}{z})}$? Prove your answer.

(3) Let D be a domain and let $f(z) = z^2 e^{\overline{z}}$ be the function $D \to \mathbb{C}$. Prove that $|f(z)|$ has no maximum value in D.

(4) Let C be the positively oriented circle of radius $\frac{1}{2}$ centered at i. Find the integral $\int_C \cot(\pi z^2) dz$.

(5) Use residues to find the improper integral $\int_0^\infty \frac{x^2}{x^4+16}dx$. Show the intermediate steps involving limits.

(6) Let C be the upper half of the clockwise oriented circle of radius 2 centered at 0. Is it true of false that $\left|\int_C \frac{z^2-1}{(z^2+1)(z^2+25)}dz\right|$ is less than $\frac{\pi}{6}$? Prove your answer. Do **not** compute the integral explicitly.

(7) Let $f(z) = e^{((z-1)^{15})}$ be the function $\mathbb{C} \to \mathbb{C}$. Use power series to find the values of the derivatives $f^{(15)}(z)$ and $f^{(23)}(z)$ at the point 1.

(8) Find all the entire functions such that $f'(z) = i$ for all z in the straight line segment from 0 to $1 + i$. Prove your answer.

(9) Is it possible to have entire functions f and g such that $(f - g)^3(x + iy) = x + iy^3$ for all $x, y \in \mathbb{R}$? If it possible find the functions, if it is not possible then prove that this is not possible.

(10) Let $f, g : \mathbb{C} \to \mathbb{C}$ be (not necessarily continuous) functions such that $|f(z)| \leq 3$ for all z and $|g(z)| < \frac{1}{\sqrt{|z|}}$ for all $z \neq 0$. Use the ϵ, δ definition of the limit to show that $\lim_{z \to \infty} f(z)g(z) = 0$.

(11) Let $z_i, i = 0, \cdots, 15$ be all the complex 16th roots of 1. Find $\sum_{i=0}^{15} z_i$. Prove your answer.