Hour Exam #1

Math 3

Oct. 10, 2012

Name (Print): ANSWER KEY
Last First

On this, the first of the two Math 3 hour-long exams in Fall 2012, and on the second hour-exam, and on the final examination I will work individually, neither giving nor receiving help, guided by the Dartmouth Academic Honor Principle.

Signature:____________________

Instructor (circle):

Lahr (Sec. 1, 8:45) Diesel (Sec. 2, 10:00)
Dorais (Sec. 3, 11:15) Dorais (Sec. 4, 12:30)
Wolff (Sec. 5, 1:45)

Instructions: You are not allowed to use calculators, books, or notes of any kind. All of your answers must be marked on the Scantron form provided or entered on the test, depending on the problem. Take a moment now to print your name and section clearly on your Scantron form and on page 1 of your exam booklet and sign the affirmation. You may write on the exam, but you will only receive credit on Scantron (multiple-choice) problems for what you write on the Scantron form. At the end of the exam, you must turn in both your Scantron form and your exam booklet. There are 15 multiple-choice problems worth 4 points each and 3 long-answer written problems worth a total of 40 points. Check to see that you have 11 pages of questions plus the cover page for a total of 12 pages.

Non-multiple choice questions:

Problem	Points	Score
16	15	
17	10	
18	15	
Total	40	

For this page, let $f(x) = \dfrac{x+4}{x^2+x-12}$.

1. What is $\lim_{x \to -\infty} f(x)$?

   (a) 0
   (b) $-\infty$
   (c) -1
   (d) The limit does not exist
   (e) None of the above

$$\lim_{x \to -\infty} \frac{x+4}{x^2+x-12} = \lim_{x \to -\infty} \frac{x+4}{x^2+x-12} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to -\infty} \frac{\frac{1}{x} + \frac{4}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2} - \frac{12}{x^2}} = \lim_{x \to -\infty} \frac{\frac{1}{x} + \frac{4}{x^2}}{1 + \frac{1}{x} - \frac{12}{x^2}} = \frac{0}{1}$$

2. What is the domain of $f(x)$?

   (a) $(-\infty, -4) \cup (-4, \infty)$
   (b) $(-\infty, \infty)$
   (c) $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
   (d) $(-\infty, -4) \cup (-4, 6) \cup (6, \infty)$
   (e) None of the above

$$f(x) = \frac{x+4}{x^2+x-12} = \frac{x+4}{(x+4)(x-3)} \Rightarrow \text{undefined at } -4, 3$$

For this page, let $f(x) = \dfrac{x^3 + 4x^2 + 4x}{(x+2)(x+3)}$.

3. What are the vertical asymptotes f?

(a) $x = -3$

(b) $x = 2$ and $x = 3$

(c) $x = 2$ and $x = -3$

(d) $y = \sqrt[3]{-8}$

(e) None of the above

undefined at -2, -3

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x(x^2+4x+4)}{(x+2)(x+3)} = \lim_{x\to -2} \frac{x(x+2)^2}{(x+2)(x+3)} = \lim_{x\to -2} \frac{x(x+2)}{(x+3)} = 0$$

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{x(x+2)}{x+3} \text{ DNE} \Rightarrow \text{vertical asymptote at } -3$$

4. What are the horizontal asymptotes of f?

(a) $y = -2$

(b) $y = 4/3$

(c) $y = 0$

(d) $y = -2$ and $y = -3$

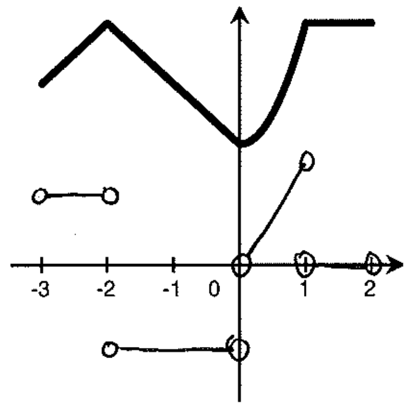
(e) None of the above

$$\lim_{x\to\infty} f(x) = \infty$$

$\Rightarrow$ no horizontal asymptote

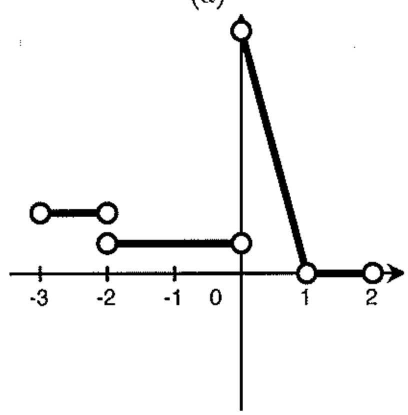
$$\lim_{x\to -\infty} f(x) = -\infty$$

5. Suppose the graph of the function $f(x)$ shown below has three linear pieces and one parabolic (polynomial of degree 2) piece

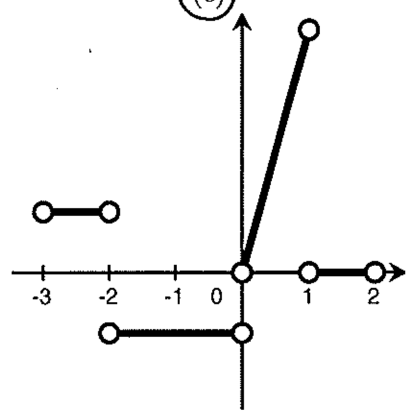


Of the following graphs, which could be the graph of $f'(x)$?

(a)



(c)



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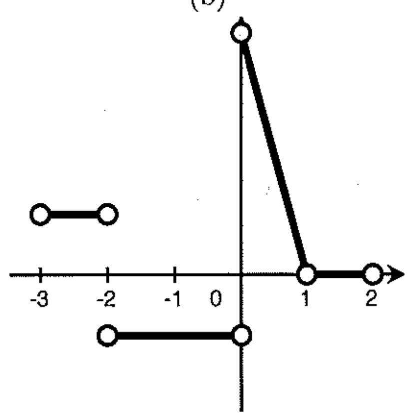
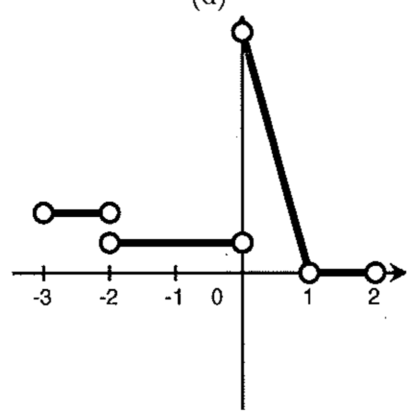
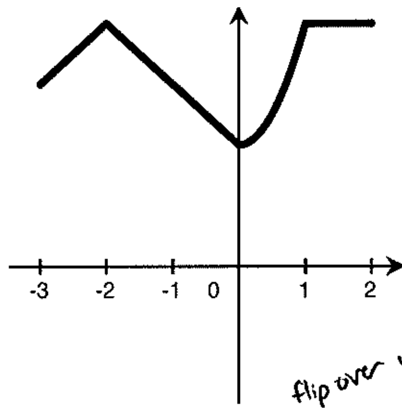


(d)



(e) None of the above

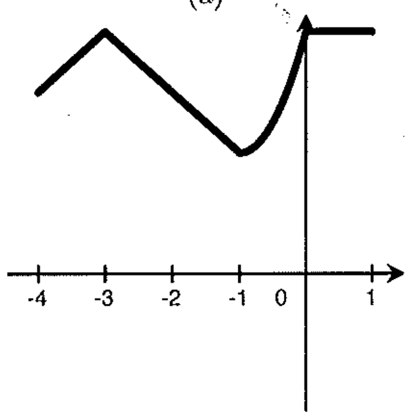
6. Again, suppose the graph of the function $f(x)$ looks like



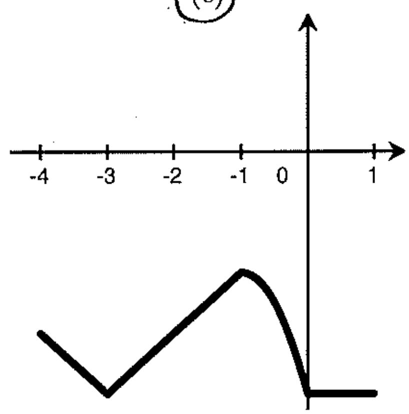
flip over y-axis
translate left by 1

Of the following graphs, which is the graph of $-f(x+1)$?

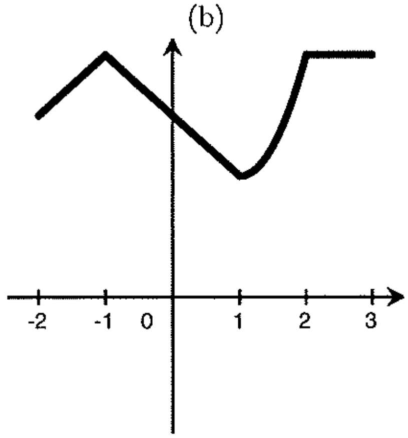
(a)



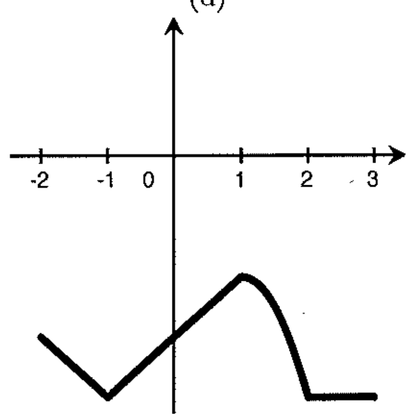
(c)



(b)



(d)



(e) None of the above

For this page, let $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{3 - \sqrt{2 - x}}$.

7. What is the domain of $f(g(x))$?

   (a) $x \leq 2$

   (b) $-7 < x \leq 2$

   (c) $2 \leq x < 11$

   (d) $x \neq 0$

   (e) $x < 2$ or $x > 11$

$$f(g(x)) = f\left(\sqrt{3-\sqrt{2-x}}\right) = \frac{1}{\sqrt{3-\sqrt{2-x}}}$$

domain of $g(x)$: $\sqrt{3-\sqrt{2-x}} \geq 0 \quad 3 \geq \sqrt{2-x} \Rightarrow 9 \geq 2-x \Rightarrow x \geq -7$

$\sqrt{2-x} \geq 0 \Rightarrow x \leq 2$

domain of $f(g(x))$: $\sqrt{3-\sqrt{2-x}} > 0$, $\sqrt{2-x} \geq 0$, $\Rightarrow \boxed{x \leq 2,\ x > -7}$

8. What is $g^{-1}(x)$?

   (a) $\sqrt{3 - \sqrt{2 - 1/x}}$

   (b) $-x^2 + 6x - 7$

   (c) $5 - x^2$

   (d) $2 - (x^2 - 3)^2$

   (e) $\dfrac{1}{\sqrt{3 - \sqrt{2 - x}}}$

$g(x) = \sqrt{3-\sqrt{2-x}}$

$x = \sqrt{3-\sqrt{2-y}}$

$x^2 = 3 - \sqrt{2-y}$

$x^2 - 3 = -\sqrt{2-y}$

$(x^2-3)^2 = 2-y$

$\boxed{y = 2-(x^2-3)^2}$

9. Let

$$f(x) = \begin{cases} x+2 & \text{if } x < -1, \\ -x & \text{if } -1 < x < 2, \\ 2-2x & \text{if } 2 \le x. \end{cases}$$

Which of the following statements about $f(x)$ is <u>false</u>?

(a) $f(x)$ is continuous on its domain ← yes since only need check at $x=2$

(b) $f(x)$ is differentiable on its domain ← NO! not diff at 2

(c) $f(x)$ has a continuous extension at $x = -1$ ← yes since $-1+2 = -(-1)$

(d) $\lim\limits_{x\to 2} f(x) = -2$ ← yes since:

(e) $f'(0) = -1$

$\overset{x+2}{\checkmark} \quad \overset{-x}{\downarrow}$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (2-2x) = -2$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (-x) = -2$

$\Rightarrow \lim\limits_{x\to 2} f(x) = -2$

yes since

$f'(0) = \frac{d}{dx}(-x)\Big|_0 = -1$

10. A particle is moving along a straight line. After 2 seconds, it is located at 3.0 m moving at a rate of 3.0 m/s. After 4 seconds, it is located at 5.0 m moving at a rate of 2.0 m/s. Which of the following necessarily happened at some point between 2 and 4 seconds?

(a) The particle was located at 2.0 m

(b) The particle was moving at a rate of 1.0 m/s

(c) The particle was moving at a rate of 0.5 m/s

(d) The particle was accelerating at a rate of $0.5\,\text{m/s}^2$

(e) The particle was accelerating at a rate of $-1.0\,\text{m/s}^2$

By MVT, ~~[illegible]~~

at some time a between 2 and 4 seconds,

$s'(a) = v(a) = \frac{ds}{dt}\Big|_a = \frac{s(4)-s(2)}{4-2} = \frac{5-3}{2} = 1\,\text{m/s}$

11. What is the range of $f^{-1}(x)$ if $f(x) = \ln(2x - 10)$?

(a) $(5, \infty)$
(b) $(0, 5)$
(c) $(-\infty, \infty)$
(d) $(-\infty, 5) \cup (5, \infty)$
(e) None of the above

range of inverse f^{-1} = domain of function f

domain of $\ln(2x-10)$: $2x - 10 > 0$

$2x > 10$

$x > 5$

12. Suppose

$$f(x) = \begin{cases} ae^x & \text{if } x > \ln 2, \\ xe^x + a & \text{if } x \leq \ln 2. \end{cases}$$

For which value of a is $f(x)$ continuous?

(a) $\ln 4$
(b) $\ln \sqrt{8}$
(c) $\ln 2$
(d) $-2 \ln 4$
(e) $\ln e^2$

at $\ln(2)$, need $ae^{\ln(2)} = \ln(2)e^{\ln(2)} + a$

$a \cdot 2 = \ln(2) \cdot 2 + a$

$a = 2 \cdot \ln(2) = \ln(2^2) = \ln(4)$

13. Suppose $f(x) = \dfrac{\tan(x)}{x^3} + 3x$. Then $f'(x)$ equals:

(a) $\dfrac{3x^2\tan(x) - x^3\sec^2(x)}{x^6}$

(b) $\dfrac{x^3\tan(x) + 3x^2\sec^2(x)}{x^6} + 3$

(c) $\dfrac{x^3\sec^2(x) - 3x^2\tan(x)}{x^6} + 3$

(d) $\dfrac{\sec^2(x)}{x^3} + 3$

(e) None of the above

$$f'(x) = \frac{x^3 \cdot (\sec^2(x)) - \tan(x)\cdot 3x^2}{x^6} + 3$$

14. Suppose $f(x) = \ln(g(x)\sqrt{x})$ where g is a differentiable function with $g(3) = \sqrt{3}$ and $g'(3) = \sqrt{3}/2$. Then $f'(3)$ equals:

(a) 2/3

(b) −2/3

(c) 1/3

(d) $2\sqrt{3}$

(e) None of the above

$$f'(x) = \frac{1}{g(x)\cdot\sqrt{x}} \cdot \left(g'(x)\sqrt{x} + \tfrac{1}{2}x^{-1/2}g(x)\right)$$

$$f'(3) = \frac{1}{\sqrt{3}\cdot\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cdot\sqrt{3} + \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\cdot\sqrt{3}\right)$$

$$= \frac{1}{3}\left(\frac{3}{2} + \frac{1}{2}\right) = \frac{1}{3}\cdot\frac{4}{2} = \frac{2}{3}$$

15. The tangent line to the curve $y^2 = xy + 4$ at $(3, 4)$ is:

(a) $-8 = 4x - 5y$

(b) $3 = -5x + 3y$

(c) $12 = x + 3y$

(d) $-5 = -x + 4y$

(e) None of the above

need $\frac{dy}{dx}$ at $(3,4)$:

$$2y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$2(4)\frac{dy}{dx} = 4 + 3\frac{dy}{dx}$$

$$8\frac{dy}{dx} = 4 + 3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4}{5}$$

$$(y-4) = \frac{4}{5}(x-3)$$

$$5(y-4) = 4(x-3)$$

$$5y - 20 = 4x - 12$$

$$5y - 4x = 8$$

$$-8 = 4x - 5y$$

Long answer questions

16. Let $f(x) = x^2 + x$.

(a) Compute the derivative of $f(x)$ using the limit definition of the derivative. Show all of your work and explain your steps to receive any credit.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2+(x+h)]-[x^2+x]}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2+x+h-x^2-x}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2+h}{h}$$

$$= \lim_{h\to 0} (2x+h+1) = 2x+1$$

(b) What is the equation of the tangent line to $f(x)$ at $(3, 12)$?

$$f'(3) = 2\cdot 3+1 = 7$$

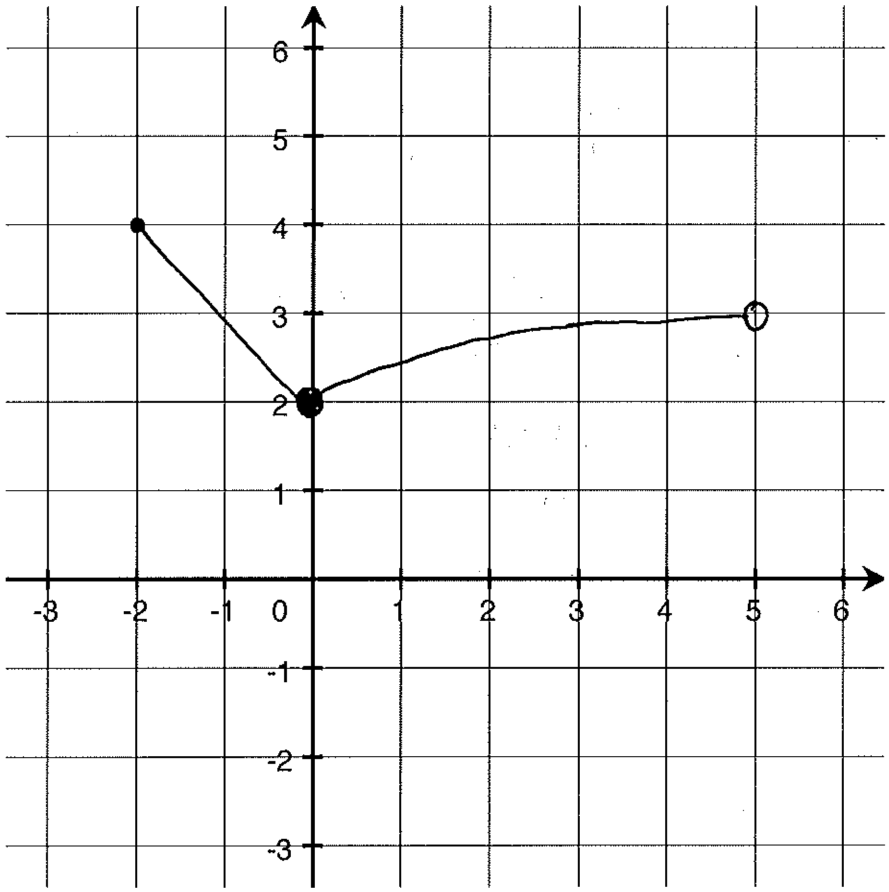
$$\boxed{y-12 = 7(x-3)}$$

or, $y = 7x-21+12$

$$\boxed{y = 7x-9}$$

17. Sketch the following function on the axes below.

$$f(x) = \begin{cases} 2 - x & \text{if } -2 \leq x < 0 \\ \sqrt{x+4} & \text{if } 0 \leq x < 5 \end{cases}$$



Is $f(x)$ differentiable at $x = 0$? Choose the best answer.

(a) Yes, because $f(x)$ is continuous at $x = 0$.

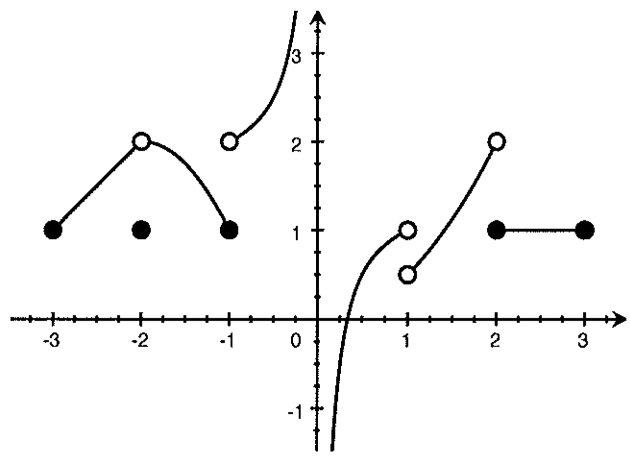
(b) Yes, because $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

(c) No, because there is a vertical tangent line at $x = 0$.

(d) No, because $f(x)$ is a piecewise defined function.

(e) No, because $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

18. Let $f(x)$ be the function graphed below.



(a) For what values of x is $f(x)$ discontinuous? List the x values only.

$x = -2, -1, 2$

(b) Evaluate the following limits:

$$\lim_{x\to -1^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = -\infty$$

$$\lim_{x\to 1^-} ((x-1)^2 f(x)) = \lim_{x\to 1^-} (x-1)^2 \cdot \lim_{x\to 1^-} f(x) = 0 \cdot 1 = 0$$

$$\lim_{x\to 2^+} (f(x)f(-x)) = \lim_{x\to 2^+} f(x) \cdot \lim_{x\to 2^+} f(-x) = 1 \cdot 2 = 2$$

(c) Where does $f(x)$ have removable discontinuities? List the x values for these discontinuities together with the corresponding y values that would remove the discontinuity there.

Only at $x=-2$.
at $x = -2$, could remove by letting $y = 2$

~~[illegible]~~