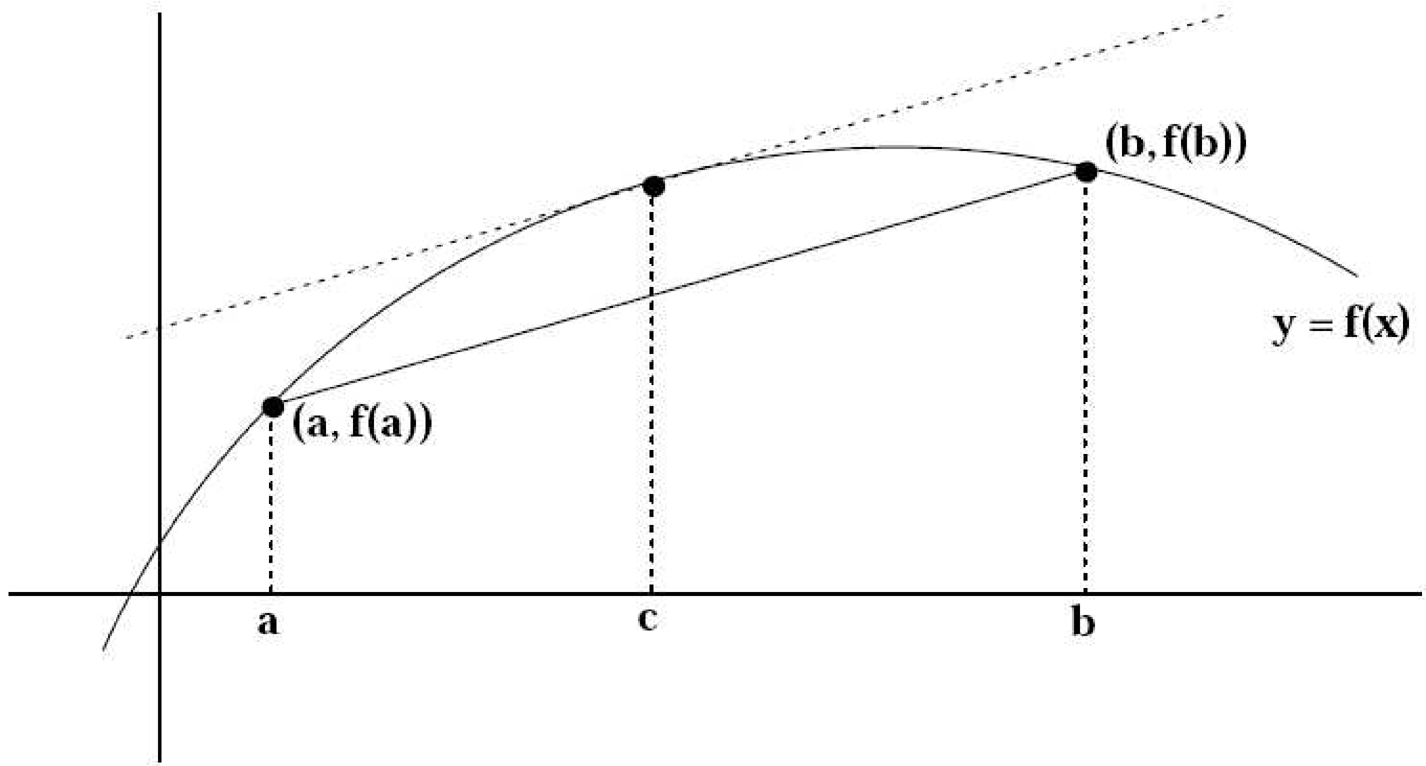
# The Mean Value Theorem

10/17/2005

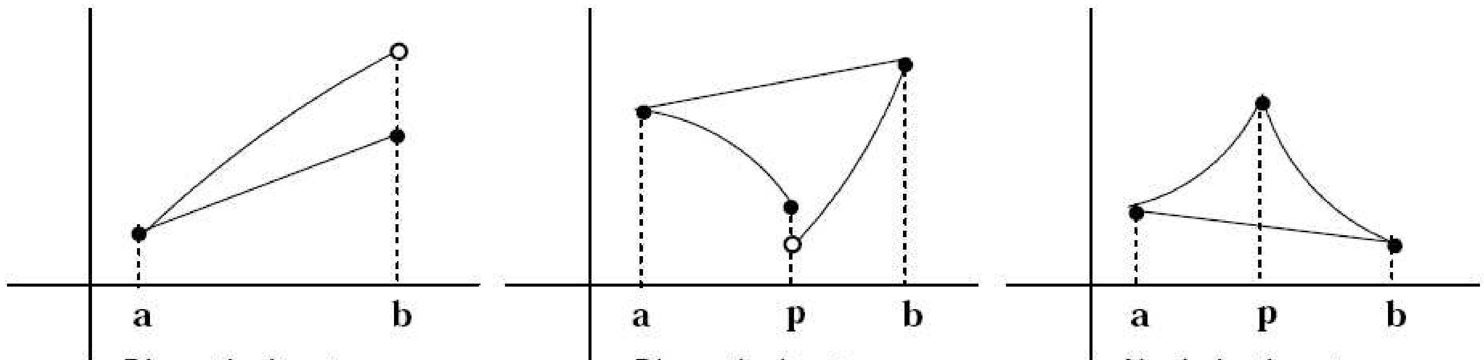
# The Mean Value Theorem

**Theorem.** *Suppose that $f$ is defined and continuous on a closed interval $[a, b]$, and suppose that $f'$ exists on the open interval $(a, b)$. Then there exists a point $c$ in $(a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$


(b, f(b))
y = f(x)
(a, f(a))
a
c
b

# Examples



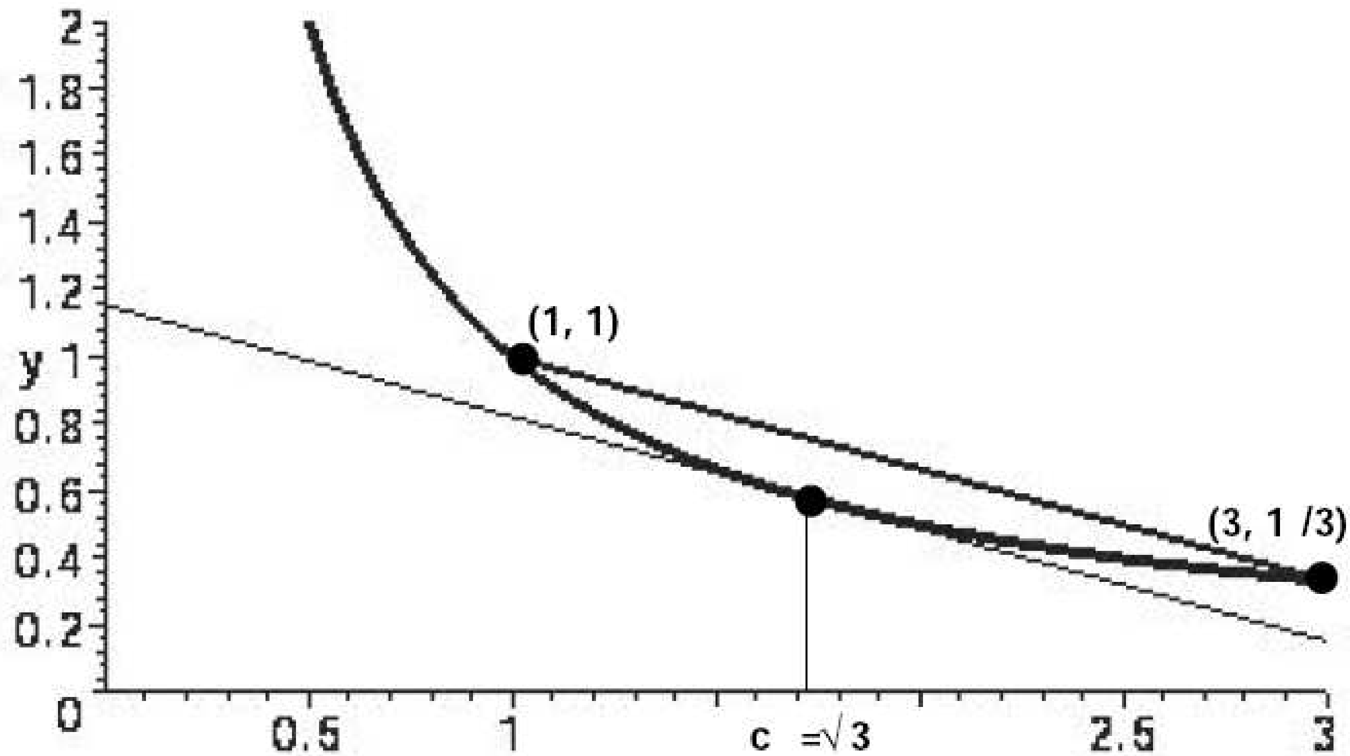

*Discontinuity at an end point*

*Discontinuity at an interior point p*

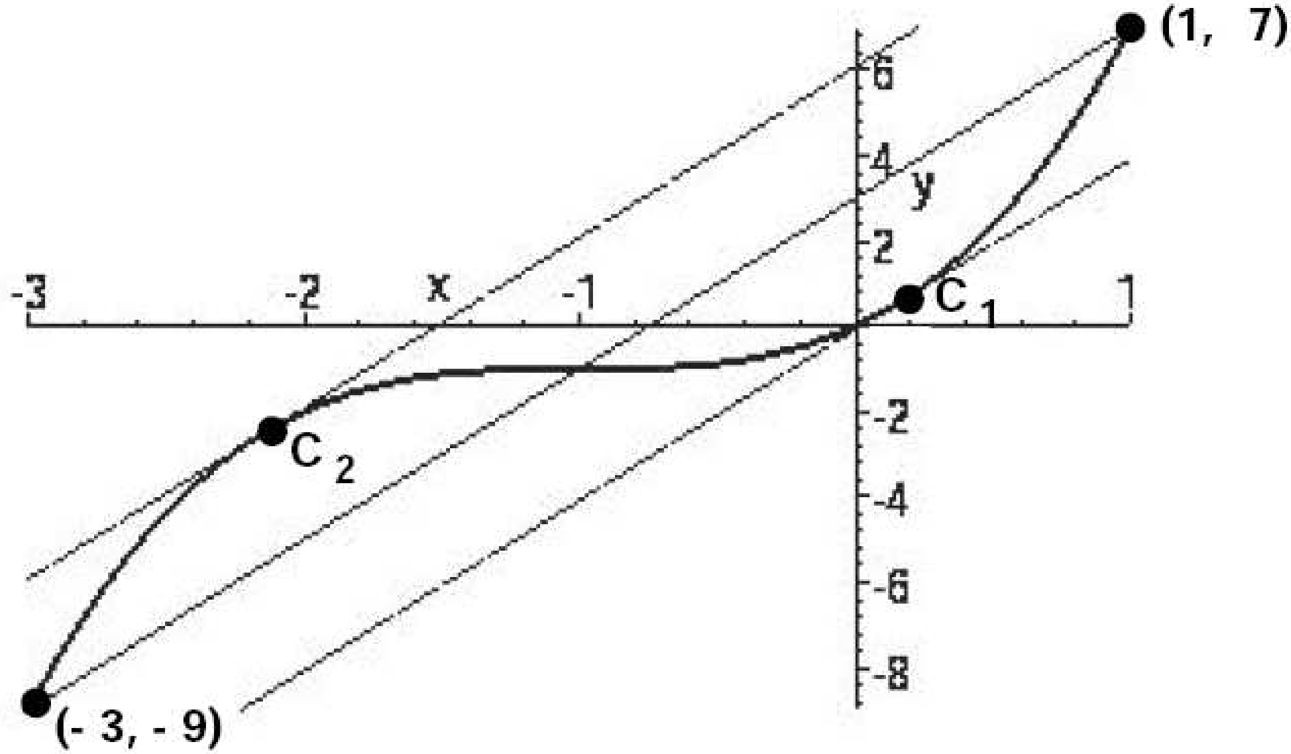
*No derivative at an interior point p*

- Consider the function $f(x) = |x|$ on $[-1, 1]$.
- The Mean Value Theorem does not apply because the derivative is not defined at $x = 0$.

- Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$?



- Verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$.



# Recall

- An interval $I$ is the set of real numbers lying between $a$ and $b$, where $a$ and $b$ are real numbers or $\pm\infty$.

# Definition

Suppose that $f$ is defined on an interval $I$, and let $x_1$ and $x_2$ denote points in $I$:

1. $f$ is *increasing* on $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

2. $f$ is *decreasing* on $I$ if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

3. $f$ is *nondecreasing* on $I$ if $f(x_1) \leq f(x_2)$ whenever $x1 < x2$.

4. $f$ is nonincreasing on $I$ if $f(x_1) \geq f(x_2)$ whenever $x1 < x2$.

**Theorem.** *Let $I$ be an interval and let $J$ be the open interval consisting of $I$ minus its endpoints (if any). Suppose that $f$ is continuous on $I$ and differentiable on $J$. Then*

1. If $f'(x) > 0$ for every $x \in J$, then $f$ is increasing on $I$.

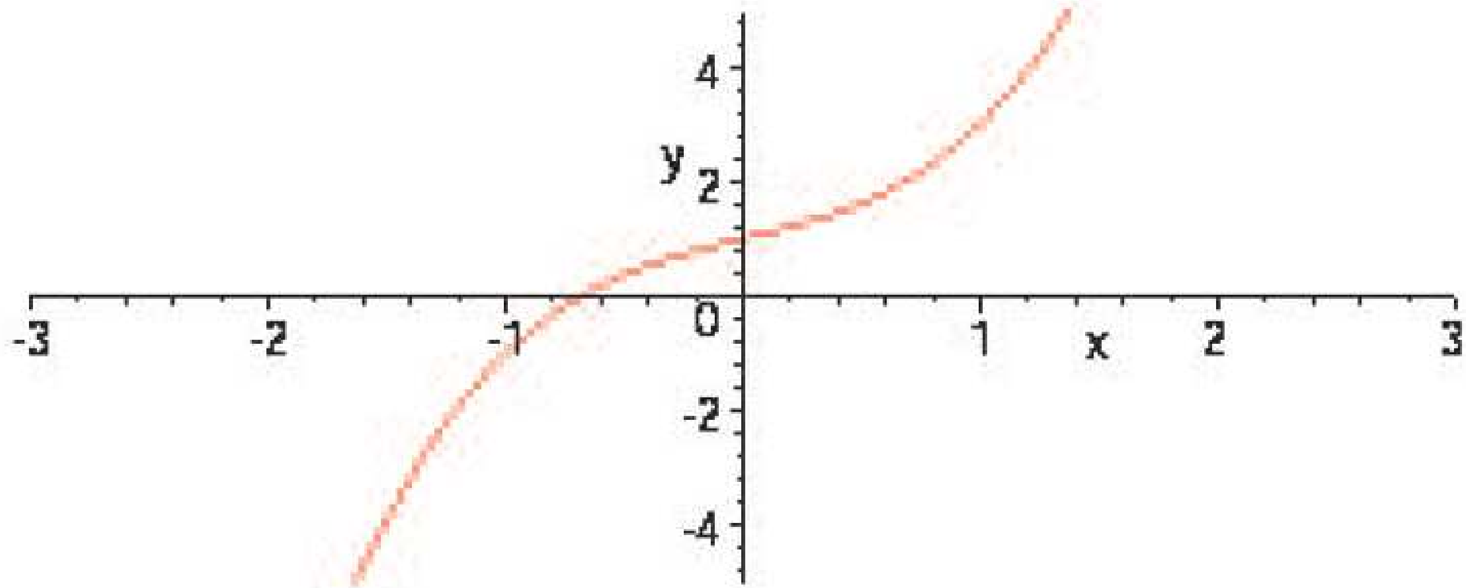
2. If $f'(x) < 0$ for every $x \in J$, then $f$ is decreasing on $I$.

3. If $f'(x) \geq 0$ for every $x \in J$, then $f$ is nondecreasing on $I$.

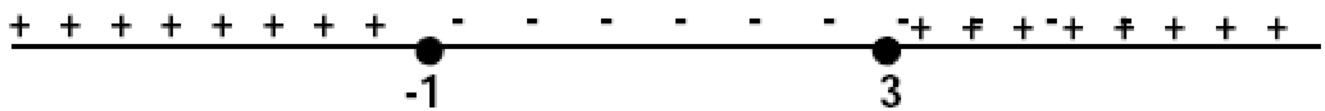
4. If $f'(x) \leq 0$ for every $x \in J$, then $f$ is nonincreasing on $I$.

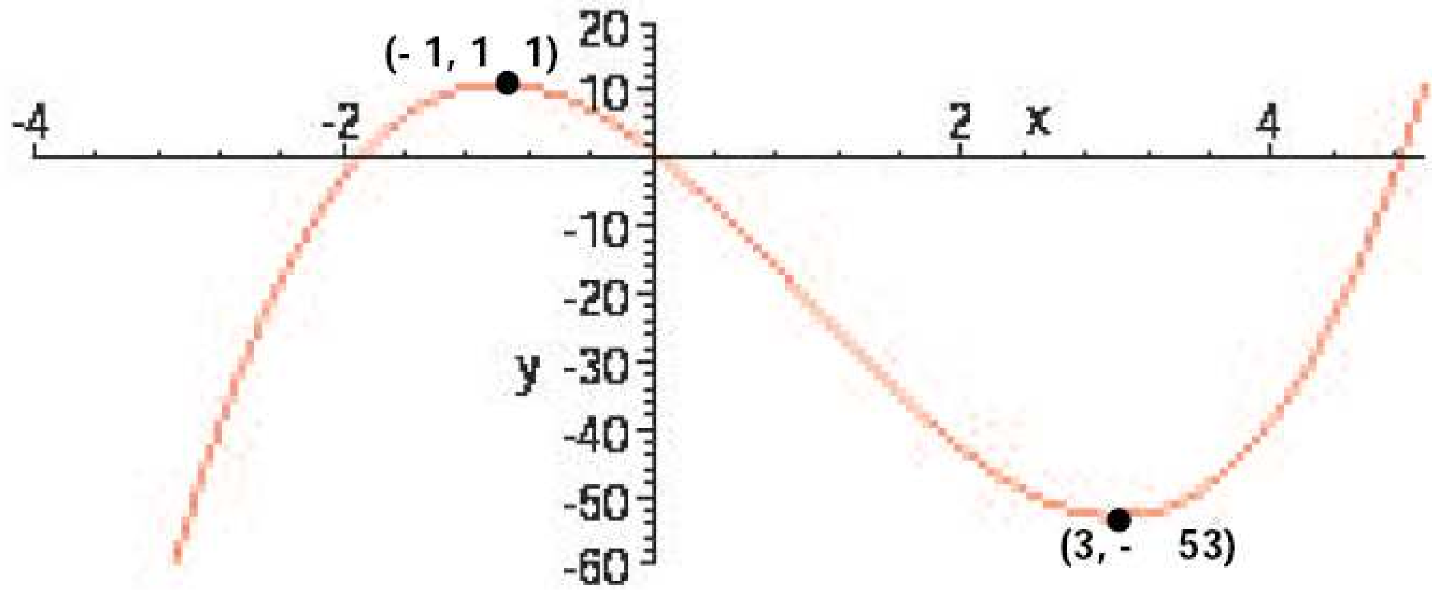
# Examples

On what interval is the function $f(x) = x^3 + x + 1$ increasing (decreasing)?



Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing.




(- 1, 1 1)
20
10
-4
-2
2
x
4
-10
-20
y
-30
-40
-50
-60
(3, - 53)

**Theorem.** *If $f$ is continuous on a closed interval $[a, b]$, then there is a point $c_1$ in the interval where f assumes its maximum value, i.e. $f(x) \leq f(c_1)$ for every $x$ in $[a, b]$, and a point $c_2$ where $f$ assumes its minimum value, i.e. $f(x) \geq f(c_2)$ for every $x$ in $[a, b]$.*

**Theorem.** *If* $f$ *is defined in an open interval* $(a, b)$ *and achieves a maximum (or minimum) value at a point* $c \in (a, b)$ *where* $f'(c)$ *exists, then* $f'(c) = 0$*.*

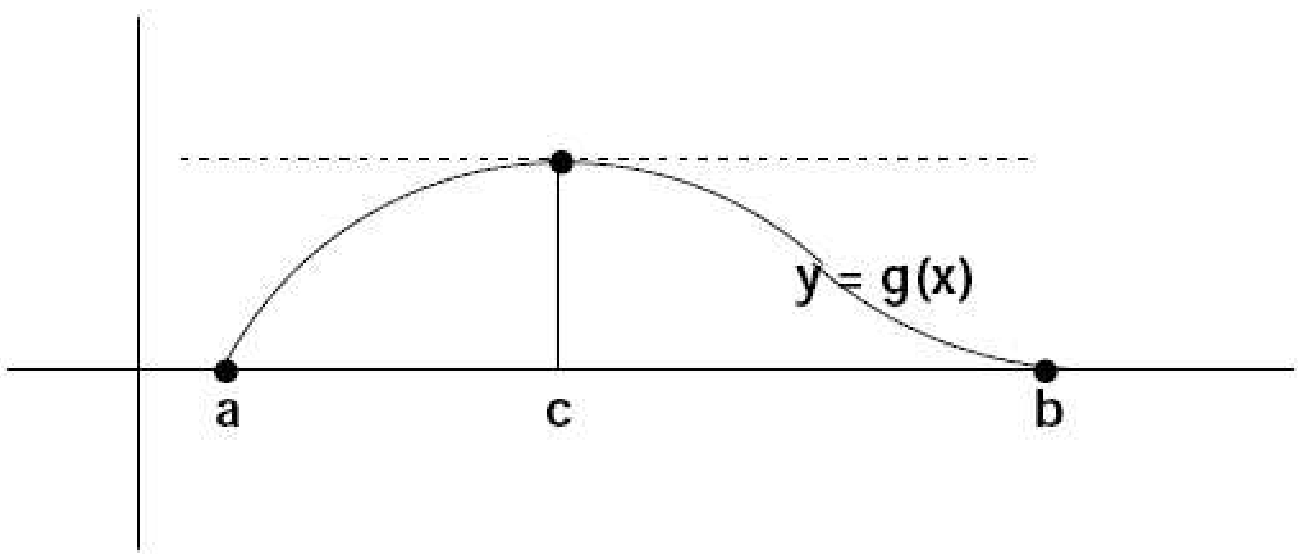
# Example

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

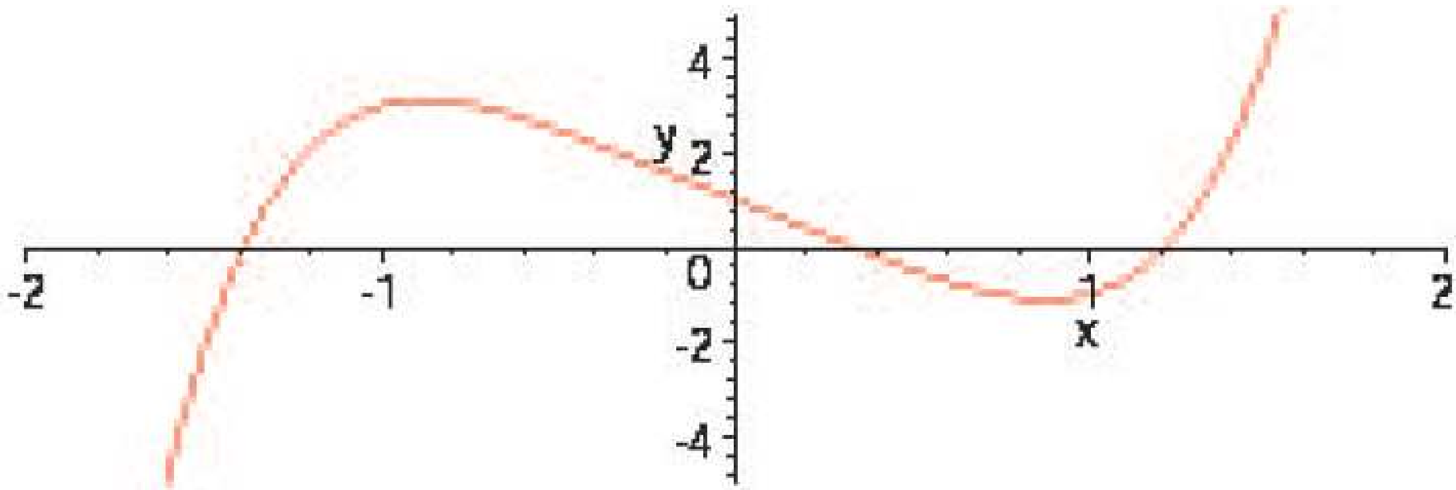
| $x$ | $f(x)$ |
|---|---|
| $-1$ | $11$ |
| $3$ | $53$ |
| $-4$ | $-151$ |
| $4$ | $-39$ |

# Rolle's Theorem

**Theorem.** *Suppose that the function $g$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. If $g(a) = 0$ and $g(b) = 0$ then there exists a point $c$ in the open interval $(a, b)$ where $g'(c) = 0$.*



Use Rolle's Theorem to show that the equation $x^5 - 3x + 1 = 0$ has exactly three real roots.



# Implicit Differentiation

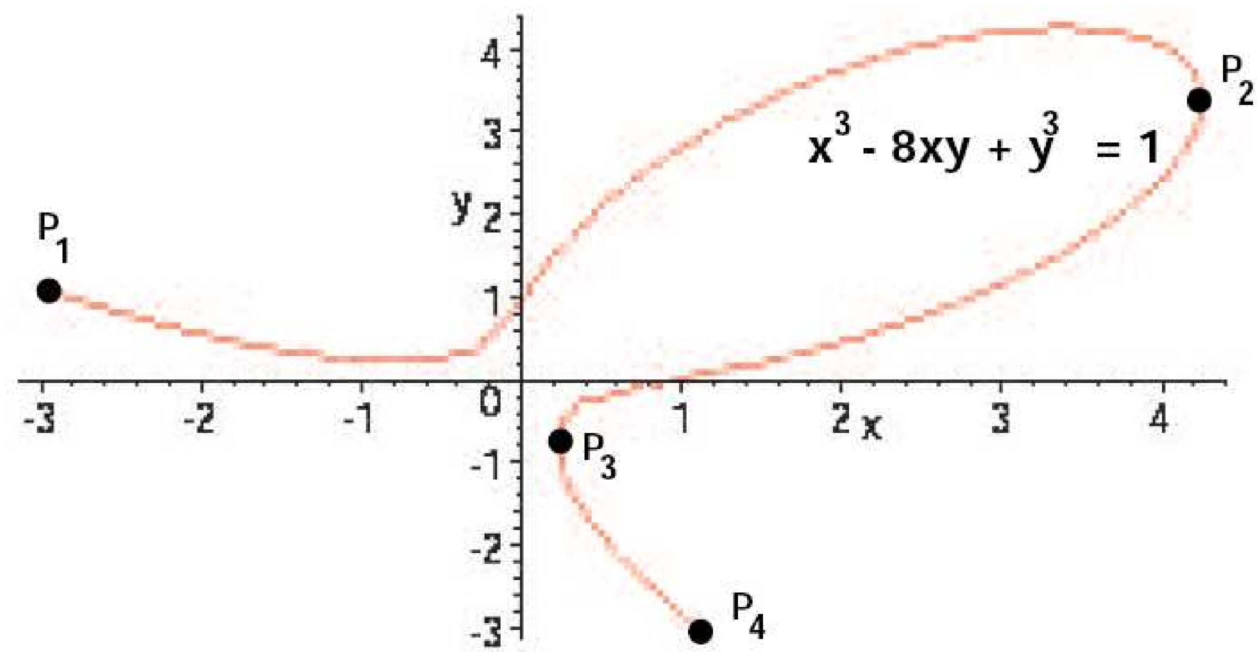
- Many curves are not the graphs of functions.

- A circle of radius 1, for example, does not pass the “vertical line test” and hence is not the graph of a function.

- It is, however, the graph of the equation $x^2 + y^2 = 1$.

- The equation $x^3 - 8xy + y^3 = 1$ resists our most clever efforts to explicitly solve for $y$ as a function of $x$.

- We will see how to overcome this difficulty using a very important technique called implicit differentiation.



- The general setting for our discussion of implicitly defined functions is an equation $F(x, y) = 0$, where $F$ is an expression containing the two variables $x$ and $y$.

- A function $f(x)$ is said to be implicitly defined by the equation if $F(x, f(x)) = 0$ on some interval $I$.

- GOAL: Find the derivative of f(x) without explicitly solving the equation.

# Examples

- The functions $\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$ are implicitly defined by the equation $x^2 + y^2 = 1$.

- Consider one of the functions $f(x)$ defined implicitly by the equation $x^2 + y^2 = 1$. Consider one of the functions f(x) defined implicitly by the equation x2 + y2 = 1.

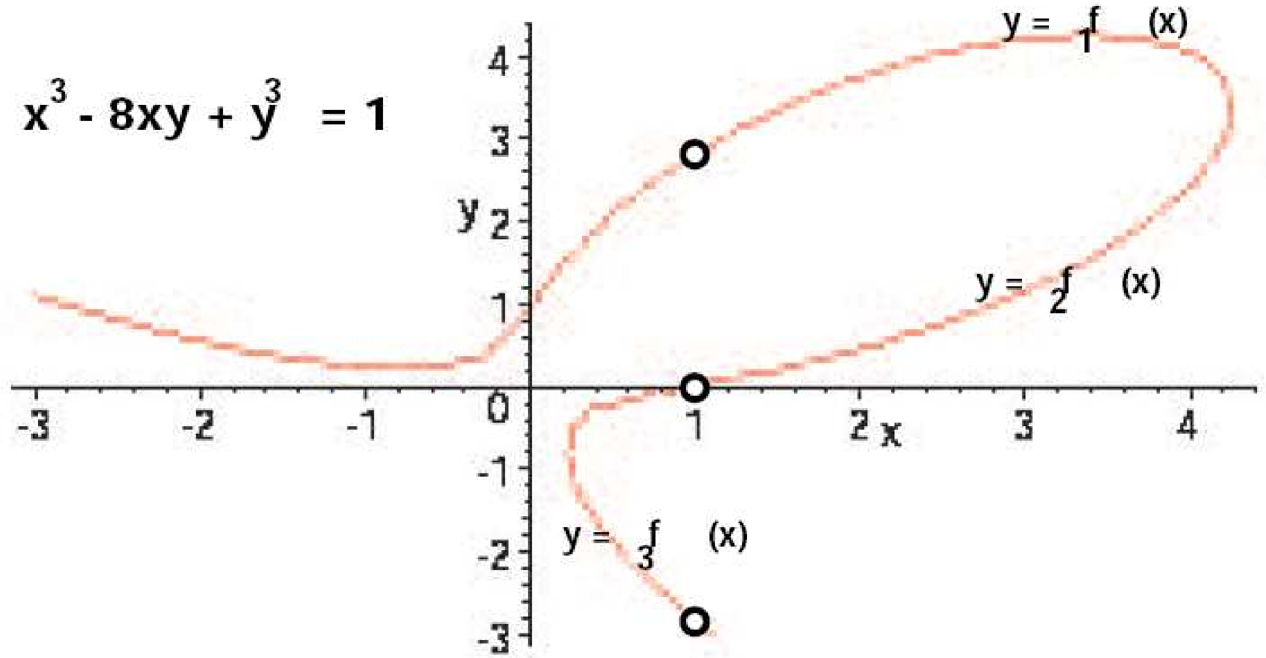
$$f'(x) = -\frac{x}{f(x)}.$$

- Given the equation $x^2 + y^2 = 1$, we think of the functions $y = f(x)$ implicitly defined by the equation.

$$\frac{dy}{dx} = -\frac{x}{y}.$$

- Use implicit differentiation to find the equation of the tangent line to the graph of $xy^2 + x^2y - 6 = 0$ at the point $(1, 2)$.

- Return to the equation $x^3 - 8xy + y^3 = 1$ with which we begin this section. Find the slope at the points on the curve for which $x = 1$.



- Suppose a differentiable function $f$ has an inverse $f^{-1}$. Find the derivative of $f^{-1}$.

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

$$\left[f^{-1}(x)\right]' = \frac{1}{f'(f^{-1}(x))}$$