Differentiation Rules

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Differentiability is Stronger than Continuity

Theorem. *If $f'(a)$ exists, then f is continuous at a.*

A function whose derivative exists at every point of an interval is not only continuous, it is *smooth*, i.e. it has no sharp corners.

Building the Toolbox

Theorem. *Suppose* $y = f(x)$ *is a function that has derivative* f'. *Then,* $(cf)' = cf'$, *where* c *is a constant. Or in Leibniz's notation* $\frac{d}{dx}(cf(x)) = c\Delta\frac{d}{dx}f(x)$.

Theorem. *If* f *and* g *are functions with derivatives* f' *and g0, respectively, then* $(f+g)' = f' + g'$. *In words, the derivative of a sum is the sum of the derivatives.*

Examples

$$\frac{d}{dx}(3x^2 + 2x + 7)$$

$$\frac{d}{dx}(x + \sqrt{x})$$

The Product Rule

Theorem. *If f and g are functions with derivatives f' and g', respectively, then $(fg)' = fg' + gf'$. In words, "the derivative of a product is the first factor times the derivative of the second, plus the second factor times the derivative of the first".*

Examples

- Find $f'(x)$ in two ways, given $f(x) = (5x + 3)(x + 2)$.

- If $y = \sqrt{x}(x^2 + 2)$, find $\frac{dy}{dx}$.

The Reciprocal of Calculus Modeling

Theorem. *Suppose f has derivative f'. Then for any x such that $f(x) \neq 0$, $\left(\frac{1}{f}\right)' = -\frac{f(x)'}{f(x)^2}$. That is, $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$.*

Example

- Find $f'(x)$ given $f(x) = \frac{1}{x^2+1}$.

The Quotient Rule

Theorem. *Suppose* f *and* g *have derivatives* f' *and* g'*, respectively. Then for any* x *such that* $g(x) \neq 0$, $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f(x)' - f(x)g(x)'}{g(x)^2}$*. That is,* $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ *. In words, "the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared".*

Examples

- Find $f'(x)$ given
$$f(x) = \frac{x+1}{x+2}.$$

- Find $f'(x)$ given
$$f(x) = \frac{1+\sqrt{x}}{x^2+3x+2}.$$

Example

- For $f(x) = \frac{1}{x} = x^{-1}$, find the derivative three ways, using the power rule, the reciprocal rule, and the quotient rule.

The Chain Rule

Theorem. *Let* $(f \circ g)(x) = f(g(x))$ *be the function defined from* f *and* g *by composition. Assume that* g *is differentiable at the point* x *and that* f *is differentiable at the point* $g(x)$. *Then the composite function* $f \circ g$ *is differentiable at the point* x, *and*

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

Using Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Examples

- Differentiate
$$f(x) = \sqrt{x^2 + 1}.$$
- Differentiate
$$y = (x^2 + 2)^{10}.$$

- Differentiate

$$f(x) = (1 + 3\sqrt{x})^{35}.$$

- Differentiate

$$f(x) = \left(\frac{x+1}{x^2+1}\right)^3.$$