The Fundamental Theorem of Calculus

We are about to discuss a theorem that relates derivatives and definite integrals. It is so important in the study of calculus that it is called the *Fundamental Theorem of Calculus*. It also gives us a practical way to evaluate many definite integrals without resorting to the limit definition. The theorem has two main parts that we will state separately as Part I and Part II.

Fundamental Theorem of Calculus (Part I-antiderivative): Suppose that f is a continuous function on the interval I containing the point a. Define the function F on I by the integral formula

$$F(x) = \int_a^x f(t)\,dt$$

Then F is differentiable on I and $F'(x) = f(x)$. That is, F is an antiderivative of f on I.

Fundamental Theorem of Calculus (Part II-evaluation): If $G(X)$ is any antiderivative of f on I (that is, $G'(x) = f(x)$ on I), then for any b in I,

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

This theorem is truly remarkable. Leibniz seems to have been the first one to recognize its generality and significance. Let's look at some examples so that we can gain a better understanding of what the theorem says, and then we will outline a proof.

Example 1: To compute $\int_0^1 (x+1)dx$, we need only find an antiderivative of $x+1$, namely, $x^2/2+x$. Then we evaluate this antiderivative at 1 and subtract its value at 0. Thus, $\int_0^1 (x+1)\,dx = (1/2+1) - (0) = 3/2$.

We normally use a vertical bar to indicate evaluation of the antiderivative at the endpoints of the interval. That is,

$$G(x)|_a^b = G(b) - G(a)$$

Example 2: $\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$.

Example 3: $\int_0^{\pi/4} \sin x\,dx = -\cos x|_0^{\pi/4} = -1/\sqrt{2} - (-1) = 1 - 1/\sqrt{2}$.

Example 4: $\int_0^{\pi/4} \sec^2 x\,dx = \tan x|_0^{\pi/4} = 1 - 0 = 1$.

We can also illustrate Part I of the Fundamental Theorem.

Example 5: $\frac{d}{dx}\int_1^x t^2\,dt = x^2$.

Example 6: $\frac{d}{dx}\int_1^{x^2} t^3\,dt = (x^2)^3 \cdot 2x$ where we first have used the Fundamental Theorem and then the chain rule to complete the calculation of the derivative.

Example 7: Consider $\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}\,dt$. We first have to put the integral in the correct form so that we can use the Fundamental Theorem:

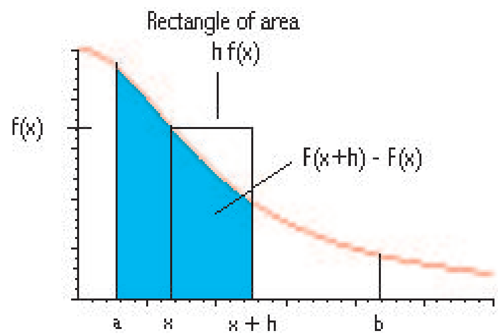
$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}\,dt &= \\
&= \frac{d}{dx}\left(\int_{x^2}^{0} e^{-t^2}\,dt + \int_0^{x^3} e^{-t^2}\,dt\right) \\
&= \frac{d}{dx}\left(-\int_0^{x^2} e^{-t^2}\,dt + \int_0^{x^3} e^{-t^2}\,dt\right) \\
&= -e^{-x^4}(2x) + e^{-x^6}(3x^2)
\end{aligned}$$

Now that we have gained some experience with the Fundamental Theorem through examples, let's look at a sketch of a proof in a special case.

Proof of the Fundamental Theorem (Part I): Fix x in I. Given that $F(x) = \int_a^x f(t)\,dt$, we need to evaluate the limit

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

But look at the sketch below. Notice that $F(x)$ is the area under the graph of f and above the interval $[a, x]$, while $F(x+h)$ is the area under the graph of f and above the interval $[a, x+h]$. Thus, $F(x+h) - F(x)$ is the area under the graph of f and above the interval $[x, x+h]$.



But for small values of h, this area is approximately equal to the area of the rectangle of height $f(x)$ on the same base; its area is length times width, or $h \cdot f(x)$. Thus, for small h, the difference quotient is approximately equal to $\frac{hf(x)}{h} = f(x)$. In other words,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

thereby completing the proof of Part I.

Proof of the Fundamental Theorem (Part II): From Part I, we have that $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f. If G is another antiderivative, then we know from a previous result that they must differ by a constant. That is, $G(x) = F(x) + C$. Now, we know that $F(a) = \int_a^a f(t)\,dt = 0$. Thus, we can determine the value of C: $G(a) = F(a) + C = 0 + C = C$. Hence, $G(x) = F(x) + G(a)$, or $F(x) = G(x) - G(a)$. So, if b is any point in I, we have $G(b) - G(a) = F(b) = \int_a^b f(t)\,dt$, which is what we wanted to prove.

Another Proof of the Fundamental Theorem of Calculus

Theorem statement: If $G(x)$ is any antiderivative of f on I (that is, $G'(x) = f(x)$ on I), then for any b in I,

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

We are going to prove this result by an application of Euler's Method which we studied earlier. Suppose we consider the Initial Value Problem

IVP: $y' = f(x), y(a) = 0, a \le x \le b$, where a, b are in I, and we want to find $y(b)$. Then because both y and G are antiderivatives of f on $[a, b]$, $y(x) = G(x) + C$ for some constant C on $[a, b]$. Then $0 = y(a) = G(a) + C$ implies $C = -G(a)$ and hence $y(b) = G(b) - G(a)$. Now, we will use Euler's method to approximate $y(b)$.

Suppose we use an integral number n of steps where each step has size $\frac{b-a}{n}$. Then, starting at the point $(a, 0)$ where the slope is $y'(a) = f(a)$, we generate the following points:

Point (x, y)	Slope
$(a, 0)$	$f(a)$
$(a + h, f(a)\ h)$	$f(a + h)$
$(a + 2h, f(a)\ h) + f(a + h)\ h)$	$f(a + 2h)$
etc.	etc.

The endpoint at $x = b$ has y-coordinate

$$\sum_{i=0}^{n-1} f(a + ih)\ h$$

The above sum is the Euler method approximate value of $y(b)$ which converges to $y(b)$ as $h \to 0$. But note that it is also a Riemann sum for the definite integral from a to b of f, and the Riemann sum converges to the value of the integral as $h \to 0$. Thus, because the limit of the sum is unique, we have

$$y(b) = \int_a^b f(x)\ dx$$

and from the result $y(b) = G(b) - G(a)$ in the first paragraph of the proof, we see that the proof is complete.

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**