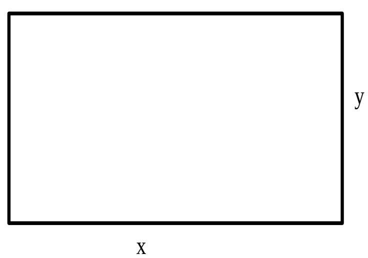
Optimization

Another application of mathematical modeling with calculus involves word problems that seek the largest or smallest value of a function on an interval. This class of problems is called *optimization* problems. For example, suppose we want to know the dimensions of a rectangle of fixed perimeter, say 1 meter, that maximizes the area. We solved this problem in the last section as an example of optimization and found that the answer is a square, $\frac{1}{4}$ meter on a side. We can outline the steps of a general procedure to follow to solve such problems, but the best way to learn is through practice.

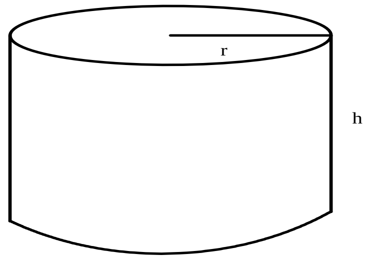
General procedure for solving optimization problems:

1. Begin by making a sketch whenever you can.
2. Define symbols and write down what is given, what is to be found.
3. Write equations that link the variables.
4. Rewrite the quantity to be maximized, or minimized, as a function of a single variable.
5. Take note of the domain over which the optimization is to occur.
6. Treat the function to be maximized, or minimized, much as you would in a sketching problem and find the extreme value(s).

Example 1: Suppose a rectangle has a fixed area of 9 square meters. Find the dimensions that minimize the perimeter. Let x and y be the lengths of the sides, as in the sketch below. Then the area is $9 = xy$, and the perimeter P is given by $P = 2x + 2y$. We want to minimize the perimeter, but it is a function of two variables. However, we can solve for y in the area equation to get $y = \frac{9}{x}$ and substitute it into the perimeter equation. This will give us P as a function of the single variable x. Here are the results: $P = 2x + \frac{18}{x}$, where $x > 0$. Taking the derivative of P and setting it equal to 0, we get: $P'(x) = 2 - \frac{18}{x^2}$; hence $P'(x) = 0$ implies $2x^2 = 18$ or $x = 3$. Using the second derivative test we see that $P''(x) = \frac{36}{x^3}$ which is positive, so $x = 3$ is indeed a minimum (local but also absolute). Thus, the perimeter will be a minimum when the rectangle is a square, 3 meters on a side.



Example 2: Find the dimensions of the 1-liter cylindrical can that can be made from the least amount of tin. This problem is of interest to can-makers who would like to minimize the cost of raw materials. However, if the solution turns out to be too tall and skinny a can, its shape might not be appropriate for the intended use. So, the can actually used in practice may cost a bit more to make than the minimal one. But let's solve the problem and see what the dimensions of the minimal can are. We shall assume that the can is a perfect cylinder without seams. The volume of the can is 1000 cubic centimeters. If r is the radius of the can and h is the height, then $\pi r^2 h = 1000$.



Moreover, the surface area of the can is the sum of the areas of the top, bottom, and side. We can calculate the area of the side by thinking of cutting it open along a line perpendicular to the top and bottom, and laying the side out on a flat surface; it is then a rectangle with one edge of length h and the other edge of length equal to the circumference of the top, namely, $2\pi r$. Thus, the side has surface area $2\pi rh$ and the top and bottom each have area πr^2. So, the total surface area of the can is $A = 2\pi rh + 2\pi r^2$.

The problem is to minimize the surface area A. But as it stands, A is a function of two variables; we need to eliminate one of them. We use the equation $\pi r^2 h = 1000$ to solve for h: $h = 1000/(\pi r^2)$. Substituting into the expression for A yields:

$$A = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2$$

The domain of A is the positive real numbers. Taking the derivative of A and setting it equal to 0 yields:

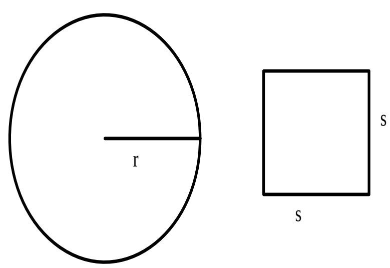
$$A' = -\frac{2000}{r^2} + 4\pi r = 0$$
$$-2000 + 4\pi r^3 = 0$$
$$\pi r^3 = 500$$
$$r = \sqrt[3]{\frac{500}{\pi}}$$

Using the Second Derivative Test, we see that $A'' = \frac{4000}{r^3} + 4\pi$; hence,

$$A''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0$$

and so A has a minimum for that value of r. Note that $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \;\; cm$ and $h = 1000/(\pi r^2) \approx 10.84 \;\; cm$ means that the can will look approximately square in profile.

Example 3: Cut a wire of length L into two pieces and bend one piece to make a square and the other to make a circle. How should you cut the wire so that the sum of the areas of the square and the circle is a minimum? To solve the problem, begin with a sketch:



Then the sum of the areas of the square and circle is $A = s^2 + \pi r^2$. And the sum of the perimeters is L: $L = 4s + 2\pi r$. Next, we substitute for s in A so that A becomes a function of the single variable r: $4s = L - 2\pi r$, or $s = (L - 2\pi r)/4$. Thus,

$$A = \left(\frac{L - 2\pi r}{4}\right)^2 + \pi r^2$$

Now, $r \geq 0$ and $L - 2\pi r \geq 0$ imply that $0 \leq r \leq \frac{L}{2\pi}$; this is the domain of A. Differentiating, we get

$$A' = 2\left(\frac{L - 2\pi r}{4}\right)\left(-\frac{2\pi}{4}\right) + 2\pi r$$

$$A' = -\frac{\pi}{4}(L - 2\pi r) + 2\pi r$$

$$A' = -\frac{\pi}{4}L + 2\pi r\left(1 + \frac{\pi}{4}\right)$$

Setting A' equal to 0, we get

$$r = \frac{\frac{\pi}{4}L}{2\pi\left(1 + \frac{\pi}{4}\right)}$$

$$r = \frac{L}{8\left(\frac{4+\pi}{4}\right)}$$

$$r = \frac{L}{2(4+\pi)}$$

Now, $A(r)$ is defined on the closed interval $\left[0, \frac{L}{2\pi}\right]$. So, to determine the extreme values of A, we need only compute the values of A at the critical point and at the endpoints, and compare them. Note that $r = 0$ corresponds to all of the wire being used for the square, and $r = \frac{L}{2\pi}$ corresponds to all of the wire being used for the circle. With $A(r) = \left(\frac{L-2\pi r}{4}\right)^2 + \pi r^2$ we have:

$$A(0) = \left(\frac{L}{4}\right)^2 = \frac{L^2}{4 \cdot 4}$$

$$A\left(\frac{L}{2\pi}\right) = 0 + \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$

$$A\left(\frac{L}{2(4+\pi)}\right) = \left(\frac{L - \frac{\pi L}{4+\pi}}{4}\right)^2 + \pi\left(\frac{L^2}{4(4+\pi)^2}\right) = \frac{L^2}{4(4+\pi)}$$

Thus, $\pi < 4 < 4 + \pi$ shows that

$$A\left(\frac{L}{2\pi}\right) > A(0) > A\left(\frac{L}{2(4+\pi)}\right)$$

Hence, A is a maximum when all of the wire is used for the circle, and A is a minimum when $r = \frac{L}{2(4+\pi)}$. Thus, to minimize the sum of the areas, cut the wire $2\pi r = \frac{\pi L}{4+\pi}$ units from one end.

Applet: Optimization **Try it!**

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**