Math 31 Fall 2005

Topics in Algebra

Take-Home Midterm Exam

Due: Friday October 28 during class.

Your Name (Please Print): ____________________________________

Instructions: This is an open book, open notes exam. You may use any printed material, including your class notes, but you **cannot** consult with your classmates or other humans. You should justify all of your answers to receive full credit.

There will be a Question and Answer session on Thursday, October 27 during the x-hour. During this time, you are welcome to come to class to ask any *general* questions. You're also welcome to come just to listen to the questions your classmates ask and what answers I give. I will also be available during my normal office hour times; however, I will save most detailed answers that may pertain to a specific problem for the Thursday session.

The Honor Principle requires that you neither give nor receive any aid on this exam.

1. Consider the following sets:

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}$$

$$H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \middle| \, x \in \mathbb{R} \right\}$$

Then G is a group under the operation of matrix multiplication.

(a) Prove that H is a subgroup of G (8 points).

(b) Is $H \trianglelefteq G$? Justify your answer (4 points).

2. Let G be a (not necessarily cyclic) group of order 44 and let $a \in G$ be an element such that $|a^5| = 11$. Prove that $|a| = 11$. (12 points)

3. Let G be a group and $a, b \in G$ such that $|a^3| = |b^3|$. Does it follow that $|a| = |b|$? If yes, provide a proof. If no, give an example supporting your answer. (8 points)

4. Consider the following two elements of S_6:

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 5 & 6 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 2 & 1 & 3 \end{bmatrix}$$

(a) Write $\alpha\beta$ as a product of disjoint cycles. (4 points)

(b) What is $|\alpha\beta|$? How do you know? (4 points)

(c) Is $\alpha\beta$ an even or odd permutation? How do you know? (4 points)

(d) What permutation is $(\alpha\beta)^{-1}$? How do you know? (4 points)

5. Suppose G is a group of order 77.

   (a) What are the possible orders for elements of G? (4 points)

   (b) Without using the general version of Cauchy's Theorem, show that G must have an element of order 7. (6 points)

(c) Suppose that G is Abelian. Prove that G is cyclic. (6 points)

6. Consider the following function:

$$\begin{aligned} \varphi : \mathbb{R} \oplus \mathbb{R} &\mapsto \mathbb{R} \\ (x, y) &\mapsto x + y \end{aligned}$$

Show that φ is a group homomorphism that is onto. (12 points)

7. Let G be a (not necessarily Abelian) group of order 108 and let $\varphi : G \mapsto \mathbb{Z}_{18}$ be a surjective homomorphism. Prove that G has a normal subgroup of order 6. (12 points)

Extra Credit: G can be shown to have normal subgroups of other orders as well. List as many orders of normal subgroups of G as you can be sure of. (3 points)

8. Let G be a non-Abelian group of order p^3 for some prime p. Assume that the center of G, $Z(G)$, is not trivial (ie., $Z(G) \neq \{e\}$). Prove that $|Z(G)| = p$. (12 points)