Practice Exam 2 Solutions

1. Compute $\int_0^{\pi/4} \cos^4 x \, dx$ Remember $\cos^2 x = \frac{1}{2}(1+\cos(2x))$

$$= \int_0^{\pi/4} \left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/4} 1 + 2\cos(2x) + \cos^2(2x) \, dx$$

$$= \frac{1}{4}\int_0^{\pi/4} 1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\, dx$$

$$= \frac{1}{4}\int_0^{\pi/4} \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\, dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\Big|_0^{\pi/4}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2}\cdot\frac{\pi}{4} + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + \frac{1}{8}\underbrace{\sin(\pi)}_{0}\right) - \underbrace{\frac{1}{4}\left(\frac{3}{2}(0) + \sin(0) + \frac{1}{8}\sin(0)\right)}_{0}$$

$$= \frac{3\pi}{32} + \frac{1}{4} = \boxed{\frac{3\pi+8}{32}}$$

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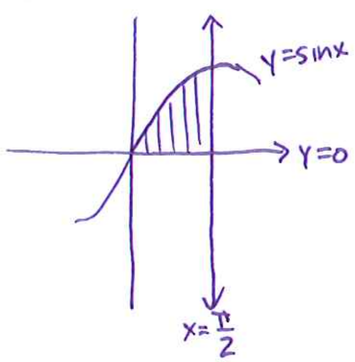
2. Volume of region bounded by $y=\sin x$, $y=0$, $x=\frac{\pi}{2}$ about y-axis.



$$\text{Volume} = \int_0^{\pi/2} 2\pi x \sin x \, dx$$

Integration by parts

$u = x$ $\quad v = -\cos x$

$du = dx$ $\quad dv = \sin x \, dx$

$$= 2\pi \int_0^{\pi/2} x \sin x \, dx$$

$$= 2\pi\left(-x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} -\cos x \, dx\right)$$

$$= 2\pi\left(-\frac{\pi}{2}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - (-0\cdot\cos(0)) + \int_0^{\pi/2}\cos x\, dx\right)$$

$$= 2\pi\int_0^{\pi/2}\cos x \, dx = 2\pi\left(\sin x\Big|_0^{\pi/2}\right)$$

$$= 2\pi\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = \boxed{2\pi}$$

3. Compute $\int \sin(3x)\sin(6x)\,dx$

$\boxed{\sin(2x) = 2\cos x \sin x}$

$$= \int \sin(3x)\cdot 2\cos(3x)\sin(3x)\,dx$$

$$= 2\int \sin^2(3x)\cos(3x)\,dx \qquad \boxed{\begin{aligned} u &= \sin(3x) \\ du &= 3\cos(3x)\,dx \end{aligned}} \quad \text{(Substitution)}$$

$$= \frac{2}{3}\int u^2\,du = \frac{2}{9}u^3 + C = \boxed{\frac{2}{9}\sin^3(3x) + C}$$

4. Compute $\int x\cos(\pi x)\,dx$

$$\begin{aligned} u &= x & v &= \tfrac{1}{\pi}\sin(\pi x) \\ du &= dx & dv &= \cos(\pi x)\,dx \end{aligned} \quad \text{(Integration by parts)}$$

$$= \frac{1}{\pi}x\sin(\pi x) - \int \frac{1}{\pi}\sin(\pi x)\,dx$$

$$= \frac{1}{\pi}x\sin(\pi x) - \frac{1}{\pi}\int \sin(\pi x)\,dx$$

$$= \frac{1}{\pi}x\sin(\pi x) - \frac{1}{\pi}\left(-\frac{1}{\pi}\cos(\pi x)\right) + C$$

$$= \boxed{\frac{1}{\pi}x\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x) + C}$$

5. (a) $\int_0^2 \sqrt{4-x^2}\,dx$

$$\begin{aligned} x &= 2\sin t \\ dx &= 2\cos t\,dt \end{aligned} \quad \text{(Trig substitution)} \qquad \begin{aligned} 0 = 2\sin t &\Rightarrow t = 0 \\ 2 = 2\sin t &\Rightarrow t = \tfrac{\pi}{2} \end{aligned}$$

$$= \int_0^{\pi/2} \sqrt{4 - 4\sin^2 t}\cdot 2\cos t\,dt$$

$$= 2\int_0^{\pi/2} \sqrt{4(1-\sin^2 t)}\,\cos t\,dt$$

$$= 2\int_0^{\pi/2} 2\cos^2 t\,dt$$

$$= 4\int_0^{\pi/2} \cos^2 t\,dt$$

$$= 4\int_0^{\pi/2} \frac{1}{2}(1+\cos(2t))\,dt$$

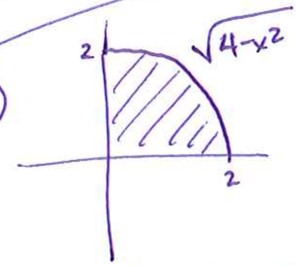
$$= 2\int_0^{\pi/2} 1 + \cos(2t)\,dt$$

$$= 2\left(t + \frac{1}{2}\sin(2t)\Big|_0^{\pi/2}\right)$$

$$= 2\left(\frac{\pi}{2} + \frac{1}{2}\cancel{\sin(\pi)}^{\,0}\right) - 2\cancel{\left(0 + \frac{1}{2}\sin(0)\right)}^{\,0}$$

$$= \boxed{\pi}$$

(b)



Quarter Circle of radius 2

$\Rightarrow$ Area $= \pi(2)^2/4 = \boxed{\pi}$

6. $\int \tan^3 x \sec^3 x \, dx$

$u = \sec x$
$du = \sec x \tan x \, dx$
(powers of tan & sec are odd!)

$= \int (\sec^2 x - 1) \tan x \sec^3 x \, dx$

$= \int (u^2 - 1) u^2 \, du = \int u^4 - u^2 \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

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7. $\int \arcsin x \, dx$

$u = \arcsin x$ $\quad v = x$
$du = \frac{1}{\sqrt{1-x^2}} dx$ $\quad dv = dx$
(Integration by parts)

$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$

$t = 1 - x^2$
$dt = -2x \, dx$
(Substitution)

$= x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{t}} dt$

$= x \arcsin x + \frac{1}{2} \int t^{-1/2} dt = x \arcsin x + \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$

$= \boxed{x \arcsin x + \sqrt{1-x^2} + C}$

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8. $\int_0^{\pi} e^{\cos(2t)} \sin t \cos t \, dt$

Recall $\sin(2t) = 2 \sin t \cos t$
$\Rightarrow \frac{1}{2}\sin(2t) = \sin t \cos t$

$= \int_0^{\pi} e^{\cos(2t)} \cdot \frac{1}{2} \cdot \sin(2t) \, dt$

$u = \cos(2t)$
$du = -2\sin(2t) \, dt$

$= -\frac{1}{4} \int_1^1 e^u \, du = \boxed{0}$

9. $\displaystyle\int_0^6 \frac{\arcsin\left(\frac{x}{6}\right)}{\sqrt{36-x^2}}\,dx$ $\quad$ $x = 6\sin t$, $dx = 6\cos t\,dt$ $\quad$ (Trig substitution) $\quad$ $0 = 6\sin t \Rightarrow t = 0$, $6 = 6\sin t \Rightarrow t = \frac{\pi}{2}$

$$= \int_0^{\pi/2} \frac{\arcsin\left(\frac{1}{6}\cdot 6\sin t\right)}{\sqrt{36-36\sin^2 t}}\cdot 6\cos t\,dt$$

$$= \int_0^{\pi/2} \frac{\arcsin(\sin t)}{\sqrt{36(1-\sin^2 t)}}\cdot 6\cos t\,dt = 6\int_0^{\pi/2} \frac{t}{6\cos t}\cdot \cos t\,dt$$

$$= \int_0^{\pi/2} t\,dt = \frac{1}{2}t^2\Big|_0^{\pi/2} = \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - (0)^2\right) = \boxed{\frac{\pi^2}{8}}$$

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10. (a) $\displaystyle\int \frac{t^5}{\sqrt{t^2+5}}\,dt$ $\quad$ Substitute $\boxed{t = \sqrt{5}\cdot\tan x}$, $dt = \sqrt{5}\cdot\sec^2 x\,dx$

$$= \int \frac{(\sqrt{5}\tan x)^5}{\sqrt{5\tan^2 x + 5}}\sqrt{5}\cdot\sec^2 x\,dx$$

$$= \int \frac{25\sqrt{5}\cdot\tan^5 x}{\sqrt{5(\tan^2 x+1)}}\cdot\sqrt{5}\cdot\sec^2 x\,dx = 25\sqrt{5}\int \frac{\tan^5 x\cdot\sec^2 x}{\sec x}\,dx$$

$$= \boxed{25\sqrt{5}\int \tan^5 x\sec x\,dx}$$

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10. (b) $\displaystyle\int \frac{1}{x^5\sqrt{9x^2-1}}\,dx$ $\quad$ Substitute $\boxed{x = \frac{1}{3}\sec t}$, $dx = \frac{1}{3}\sec t\cdot\tan t\,dt$

$$= \int \frac{1}{\left(\frac{1}{3}\sec t\right)^5\sqrt{9\left(\frac{1}{9}\right)\sec^2 t - 1}}\cdot\frac{1}{3}\sec t\cdot\tan t\,dt$$

$$= \frac{1}{3}\int \frac{\sec t\cdot\tan t}{\left(\frac{1}{3}\right)^5\sec^5 t\cdot\tan t}\,dt = \boxed{3^4\int \cos^4 t\,dt}$$

10. (c) $\int \frac{1}{x\sqrt{5-x^2}}\,dx$

$\boxed{x = \sqrt{5}\cdot \sin t}$
$dx = \sqrt{5}\cdot \cos t\,dt$

$$\int \frac{\cancel{\sqrt{5}}\cdot \cos t}{\cancel{\sqrt{5}}\cdot \sin t\sqrt{5-5\sin^2 t}}\,dt = \int \frac{\cos t}{\sin t\sqrt{5}\cdot \cos t} = \boxed{\frac{1}{\sqrt{5}}\int \csc t\,dt}$$

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11. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

$x = \sec t$
$dx = \sec t\cdot \tan t\,dt$ (Trig substitution)

$1 = \sec t = \frac{1}{\cos t} \Rightarrow \cos t = 1 \Rightarrow t = 0$
$2 = \sec t = \frac{1}{\cos t} \Rightarrow \cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}$

$$\int_0^{\pi/3} \frac{\sqrt{\sec^2 t - 1}}{\cancel{\sec t}}\cdot \cancel{\sec t}\cdot \tan t\,dt$$

$$\int_0^{\pi/3} \tan^2 t\,dt = \int_0^{\pi/3} \sec^2 t - 1\,dt = \tan t - t\Big|_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - (\tan(0) - 0)$$

$$= \frac{\sin(\pi/3)}{\cos(\pi/3)} - \frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} - \frac{\pi}{3} = \boxed{\sqrt{3} - \frac{\pi}{3}}$$

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12. Average of $f(x) = x\sin(x^2)$ on $[\sqrt{\pi}, 2\sqrt{\pi}]$.

$$\text{Avg} = \frac{1}{2\sqrt{\pi}-\sqrt{\pi}}\int_{\sqrt{\pi}}^{2\sqrt{\pi}} x\sin(x^2)\,dx$$

$u = x^2$
$du = 2x\,dx$ (Substitution)

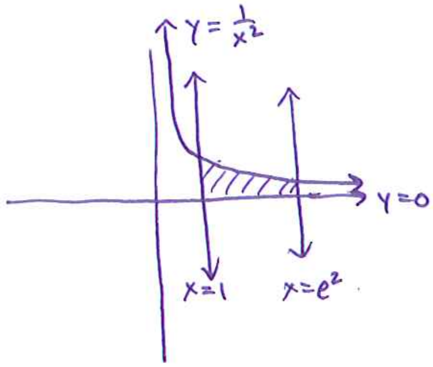
$$= \frac{1}{\sqrt{\pi}}\int_{\pi}^{4\pi} \frac{1}{2}\sin u\,du$$

$$= \frac{1}{2\sqrt{\pi}}\int_{\pi}^{4\pi} \sin u\,du = \frac{1}{2\sqrt{\pi}}\left(-\cos u\Big|_{\pi}^{4\pi}\right)$$

$$= \frac{-1}{2\sqrt{\pi}}\left(\cos(4\pi) - \cos(\pi)\right)$$

$$= \frac{-1}{2\sqrt{\pi}}\left(1 - (-1)\right) = \boxed{\frac{-1}{\sqrt{\pi}}}$$

13. Volume of region by $y = \frac{1}{x^2}$, $y=0$, $x=1$, $x=e^2$ about y-axis.



$$\text{Volume} = \int_1^{e^2} 2\pi x \left(\frac{1}{x^2}\right) dx$$

$$= 2\pi \int_1^{e^2} \frac{1}{x}\, dx$$

$$= 2\pi \left(\ln|x| \Big|_1^{e^2} \right)$$

$$= 2\pi \left(\ln(e^2) - \ln(1) \right)$$

$$= 2\pi (2-0) = \boxed{4\pi}$$

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14. $\displaystyle\int \frac{\sqrt{\arctan x}}{1+x^2}\, dx$

$x = \tan t$
$dx = \sec^2 t\, dt$

(Trig substitution)

$t = \tan^{-1} x$

$$= \int \frac{\sqrt{\arctan(\tan t)}}{1+\tan^2 t} \cdot \sec^2 t\, dt$$

$$= \int \frac{\sqrt{t}}{\sec^2 t} \cdot \sec^2 t\, dt = \int t^{1/2}\, dt = \frac{t^{3/2}}{3/2} + C$$

$$= \frac{2}{3} \tan^{3/2} t + C = \boxed{\frac{2}{3}(\arctan x)^{3/2} + C}$$

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15. Average of $f(x) = \frac{1}{x}$ on $[1,3]$.

$$\text{Avg} = \frac{1}{3-1} \int_1^3 \frac{1}{x}\, dx = \frac{1}{2}\left(\ln|x| \Big|_1^3 \right) = \frac{1}{2}\left(\ln(3) - \ln(1) \right) = \boxed{\frac{1}{2}\ln(3)} = \boxed{\ln(\sqrt{3})}$$

16. $\int \cos^2 x \sin(2x)\,dx$ $\qquad$ $\underline{\sin(2x) = 2\cos x \sin x}$

$= \int \cos^2 x \cdot 2\sin x \cdot \cos x\, dx$

$= 2\int \cos^3 x \cdot \sin x\, dx$ $\qquad$ $u = \cos x$, $du = -\sin x\, dx$ $\qquad$ (Substitution)

$= -2\int u^3\, du = -\frac{1}{2}u^4 + C = \boxed{-\frac{1}{2}\cos^4 x + C}$

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17. $\int_0^2 x^2\sqrt{4-x^2}\,dx$ $\qquad$ $x = 2\sin t$, $dx = 2\cos t\, dt$ $\qquad$ (Trig substitution) $\qquad$ $0 = 2\sin t \Rightarrow t = 0$, $2 = 2\sin t \Rightarrow t = \frac{\pi}{2}$

$= \int_0^{\pi/2} 4\sin^2 t\sqrt{4-4\sin^2 t}\cdot 2\cos t\, dt$

$= 8\int_0^{\pi/2} \cos t \cdot \sin^2 t \cdot \sqrt{4(1-\sin^2 t)}\, dt$

$= 16\int_0^{\pi/2} \cos^2 t \cdot \sin^2 t\, dt$

$= 16\int_0^{\pi/2} \left(\frac{1}{2}(1+\cos(2t))\right)\cdot\left(\frac{1}{2}(1-\cos(2t))\right) dt$

Recall that

$$\left[\begin{array}{l}\cos^2 x = \frac{1}{2}(1+\cos(2x)) \\ \sin^2 x = \frac{1}{2}(1-\cos(2x))\end{array}\right.$$

$= 16\int_0^{\pi/2} \frac{1}{4}\left(1-\cos(2t)+\cos(2t)-\cos^2(2t)\right) dt$

$= 4\int_0^{\pi/2} 1-\cos^2(2t)\, dt$

$= 4\int_0^{\pi/2} 1-\frac{1}{2}(1+\cos(4t))\,dt$

$= 4\int_0^{\pi/2} 1-\frac{1}{2}-\frac{1}{2}\cos(4t)\, dt$

$= 4\int_0^{\pi/2} \frac{1}{2}-\frac{1}{2}\cos(4t)\,dt$

$= 2\int_0^{\pi/2} 1-\cos(4t)\,dt$

$= 2\left(t - \frac{1}{4}\sin(4t)\Big|_0^{\pi/2}\right)$

$= 2\left(\frac{\pi}{2} - \frac{1}{4}\cancel{\sin(2\pi)}^{0}\right) - 2\cancel{\left(0-\frac{1}{4}\sin(4.0)\right)}^{0}$

$= \boxed{\pi}$

18. $\int_0^1 (x^2+1)e^{-x}\,dx$ $\quad u = x^2+1 \quad v = -e^{-x}$ $\quad du = 2x\,dx \quad dv = e^{-x}\,dx$ (Int. by parts)

$= -(x^2+1)e^{-x}\Big|_0^1 + 2\int_0^1 x e^{-x}\,dx$

$= -(1^2+1)e^{-1} - \left(-(0^2+1)e^{-0}\right) + 2\int_0^1 x e^{-x}\,dx$ $\quad u = x \quad v = -e^{-x}$ $\quad du = dx \quad dv = e^{-x}\,dx$ (Int. by parts again.)

$= \frac{-2}{e} + 1 + 2\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right)$

$= \frac{-2}{e} + 1 + 2\left(-1e^{-1} - (-0\cdot e^{0}) + \int_0^1 e^{-x}\,dx\right)$

$= \frac{-2}{e} + 1 - \frac{2}{e} + 2\int_0^1 e^{-x}\,dx$

$= \cancel{-\frac{4}{e}+1-2e^{-x}\Big|_0^1}$ $= -\frac{4}{e} + 1 - 2\left(e^{-x}\Big|_0^1\right)$ $= \cancel{-\frac{4}{e}-\frac{2}{e}+1}$

$= -\frac{4}{e} + 1 - 2\left(\frac{1}{e} - 1\right)$ $= -\frac{4}{e} + 1 - \frac{2}{e} + 2$ $= -\frac{6}{e} + 3 = \boxed{\frac{3e-6}{e}}$

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19. $\int_0^{\pi/4} \tan^4 x\,dx$

$= \int_0^{\pi/4} (\sec^2 x - 1)\tan^2 x\,dx$

$= \underbrace{\int_0^{\pi/4} \tan^2 x \sec^2 x\,dx}_{\substack{u = \tan x \\ du = \sec^2 x\,dx}} - \int_0^{\pi/4} \tan^2 x\,dx$

$= \int_0^1 u^2\,du - \int_0^{\pi/4} \sec^2 x - 1\,dx$

$= \frac{1}{3}u^3\Big|_0^1 - \left(\tan x - t\Big|_0^{\pi/4}\right)$

$= \frac{1}{3} - \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - (\tan(0) - 0)\right) = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4} - \frac{2}{3}}$

20. $\int_1^4 \frac{\ln|x|}{\sqrt{x}}\,dx$ $\quad u=\ln|x| \quad v=2\sqrt{x}$ $\quad du=\frac{1}{x}dx \quad dv=x^{-1/2}\,dx$ (Int. by parts)

$= 2\sqrt{x}\cdot\ln|x|\Big|_1^4 - 2\int_1^4 \frac{\sqrt{x}}{x}\,dx$

$= 2\sqrt{4}\cdot\ln(4) - 2\sqrt{1}\cdot\cancel{\ln(1)}^{\,0} - 2\int_1^4 x^{-1/2}\,dx$

$= 4\cdot\ln 4 - 2\int_1^4 x^{-1/2}\,dx$

$= 4\ln 4 - 2\left(\frac{x^{1/2}}{1/2}\Big|_1^4\right) = 4\ln 4 - 4\left(\sqrt{4}-\sqrt{1}\right) = 4\ln 4 - 4 = \boxed{4(\ln 4 - 1)}$

21. (a) $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$ $\quad x = 2\sin t$ $\quad dx = 2\cos t\,dt$ (Trig. Substitution) $\quad t = \arcsin\left(\frac{x}{2}\right)$

$= \int \frac{4\sin^2 t}{(4-4\sin^2 t)^{3/2}}\cdot 2\cos t\,dt$

$= 8\int \frac{\sin^2 t\cdot\cos t}{(4(1-\sin^2 t))^{3/2}}\,dt$

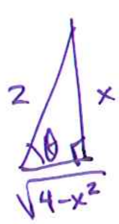
$= 8\int \frac{\cos t\cdot\sin^2 t}{8\cos^3 t}\,dt = \int \frac{\sin^2 t}{\cos^2 t}\,dt = \int \tan^2 t\,dt$

$= \int \sec^2 t - 1\,dt = \tan t - t + C$

$= \boxed{\tan\left(\arcsin\left(\frac{x}{2}\right)\right) - \arcsin\left(\frac{x}{2}\right) + C}$

21.(b) $\underbrace{\tan\left(\arcsin\left(\frac{x}{2}\right)\right)}_{\theta}$

$\theta \longleftrightarrow \sin\theta = \frac{x}{2} \longrightarrow$



$\Rightarrow \tan\left(\arcsin\left(\frac{x}{2}\right)\right) = \tan\theta = \frac{O}{A} = \boxed{\frac{x}{\sqrt{4-x^2}}}$

21. (c) $\int_0^1 \frac{x^2}{(4-x^2)^{3/2}}\,dx = F(1)-F(0)$ by FTC2, where

$F(x) = \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right)$ by part (a) & (b).

$F(1) = \frac{1}{\sqrt{4-1^2}} - \arcsin\left(\frac{1}{2}\right) = \frac{1}{\sqrt{3}} - \frac{\pi}{6}$

$F(0) = \frac{0}{\sqrt{4}} - \arcsin(0) = 0$

$\Rightarrow \int_0^1 \frac{x^2}{(4-x^2)^{3/2}}\,dx = \frac{1}{\sqrt{3}} - \frac{\pi}{6} = \boxed{\frac{2\sqrt{3}-\pi}{6}}$

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22. $\int \sec^6 x\,dx = \int (\tan^2 x+1)^2 \sec^2 x\,dx$

$u = \tan x$
$du = \sec^2 x\,dx$ (Substitution)

$= \int (u^2+1)^2\,du$

$= \int u^4 + 2u^2 + 1\,du$

$= \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C$

$= \boxed{\frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C}$