Practice Final Exam solutions

① Find the derivative of $\int_2^{x^2} \cot(t)\,dt$.

By the first Fundamental Theorem of Calculus, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. If we have some function $g(x)$ instead, $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$. For this example,

$$\frac{d}{dx}\int_2^{x^2} \cot(t)\,dt = \cot(x^2)\cdot\frac{d}{dx}(x^2) = \boxed{2x\cot(x^2)}$$

② Find $\int_1^2 x^4\,dx$.

By the second Fundamental Theorem of Calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

In this instance, since $\frac{x^5}{5}$ is an antiderivative of x^4,

$$\int_1^2 x^4\,dx = \frac{x^5}{5}\Big|_1^2 = \frac{2^5}{5} - \frac{1^5}{5} = \frac{32}{5} - \frac{1}{5} = \boxed{\frac{31}{5}}$$

③ Let $F(x) = e^{x^2}$. Evaluate $\int_{-2}^1 F'(x)\,dx$.

Since $F(x)$ is clearly an antiderivative of $F'(x)$,

$$\int_{-2}^1 F'(x)\,dx = F(x)\Big|_{-2}^1 = F(1) - F(-2) = e^{1^2} - e^{(-2)^2} = \boxed{e - e^4}$$

④ Let $F(x) = \int \sin x \cos x\,dx$. Find the formula for $F(x)$ given that $F(\pi/2) = 2$.

$F(x) = \int \sin x \cos x\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$= \int u\,du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$

$F(\pi/2) = \frac{\sin^2(\pi/2)}{2} + C = 2$

$\frac{1}{2} + C = 2$

$C = 3/2$

$F(x) = \boxed{\frac{1}{2}\sin^2 x + 3/2}$

$= 2 - \frac{1}{2}\cos^2 x$

⑤ Find $\int \frac{\sin x}{1+\cos^2 x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{1}{1+u^2}(-du)$

$= -\int \frac{1}{1+u^2}\,du = -\arctan u + C = \boxed{-\arctan(\cos x) + C}$

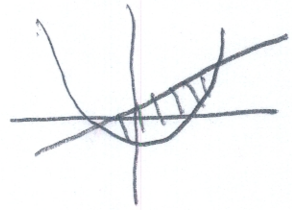
⑥ Find the area between the curves $y = x^2 - 2$, $y = x$.

Intersection: $x^2 - 2 = x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1, 2$



Testing at $x=0$, $y=x$ is above $y = x^2-2$

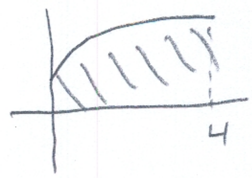
$A = \int_{-1}^{2} x - (x^2-2)\,dx = \int_{-1}^{2} x - x^2 + 2\,dx = \frac{x^2}{2} - \frac{x^3}{3} + 2x \Big|_{-1}^{2}$

$= \left[\frac{4}{2} - \frac{8}{3} + 4\right] - \left[\frac{1}{2} + \frac{1}{3} - 2\right] = \frac{12}{6} - \frac{16}{6} + \frac{24}{6} - \frac{3}{6} - \frac{2}{6} + \frac{12}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$

⑦ Use disks to find the volume of the solid obtained by rotating the region bounded by $y = 1+\sqrt{x}$, $x=4$, $x=0$, and $y=0$ about the x-axis.

$r = 1+\sqrt{x}$

$V = \pi \int_0^4 (1+\sqrt{x})^2\,dx = \pi \int_0^4 1 + 2\sqrt{x} + x\,dx$



$= \pi\left[x + \frac{4}{3}x^{3/2} + \frac{x^2}{2}\right]_0^4 = \pi\left(4 + \frac{4}{3}(8) + \frac{4^2}{2}\right)$

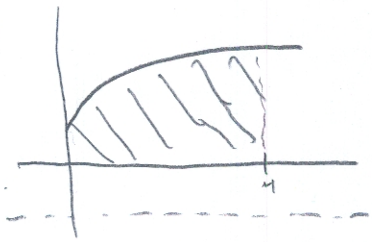
$= \pi\left(4 + \frac{32}{3} + \frac{16}{2}\right)$

$= \pi\left(\frac{24}{6} + \frac{64}{6} + \frac{48}{6}\right) = \frac{136\pi}{6} = \boxed{\frac{68\pi}{3}}$

⑧ Use washers to find the volume of the solid obtained by rotating about the line $y=-1$ the region bounded by $y=1+\sqrt{x}$, $x=0$, $x=4$, and $y=0$

$r_o = 1+\sqrt{x} - (-1) = 2+\sqrt{x}$

$r_i = 0 - (-1) = 1$



$$V = \pi\int_0^4 (2+\sqrt{x})^2 - 1^2 \, dx$$

$$= \pi\int_0^4 4+4\sqrt{x}+x-1 \, dx$$

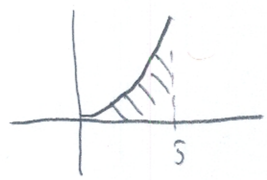
$$= \pi\int_0^4 3+4\sqrt{x}+x \, dx$$

$$= \pi\left[3x + \tfrac{8}{3}x^{3/2} + \tfrac{x^2}{2}\right]_0^4 = \pi\left(12 + \tfrac{8}{3}(8) + \tfrac{16}{2}\right) = \pi\left[\tfrac{72}{6} + \tfrac{128}{6} + \tfrac{48}{6}\right] = \frac{248\pi}{6} = \boxed{\frac{124\pi}{3}}$$

⑨ Use cylindrical shells to find the volume of the solid obtained by rotating about the y-axis the region bounded by $y=e^x$, $x=5$, $x=0$, and $y=0$.

$r = x$

$h = e^x$



$$V = 2\pi\int_0^5 xe^x \, dx$$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

using integration by parts

$$= 2\pi\left[xe^x - \int e^x dx\right]_0^5$$

$$= 2\pi\left[xe^x - e^x\right]_0^5 = 2\pi(5e^5 - e^5 - (0-1)) = \boxed{2\pi(4e^5+1)}$$

⑩ A spring has natural length 10 cm. If a 30N force is required to keep it stretched to a length of 15 cm, how much work is required to stretch it from 10 cm to 13 cm?

$F = kx$

$30 = k(.15 - .1) = .05k$

$600 \text{ N/m} = k$

$$W = \int F(x)\,dx = \int_0^{.03} 600x \, dx = 300x^2\Big|_0^{.03} = 300\left(\tfrac{3}{100}\right)^2$$

$$= 300 \cdot \tfrac{9}{10000} = \tfrac{27}{100} = \boxed{.27 \text{ J}}$$

(11) Find the average value of the function $h(x)=\frac{2}{x^2-1}$ on the interval $[3,4]$.

$$h_{ave}=\frac{1}{4-3}\int_3^4 \frac{2}{x^2-1}\,dx=\int_3^4 \frac{2}{x^2-1}\,dx$$

Partial fractions:

$$\frac{2}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}$$

$$2=A(x+1)+B(x-1)$$

x: $0=A+B$

1: $2=A-B$

Adding: $2=2A$

$A=1,\ B=-1$

$$h_{ave}=\int_3^4 \frac{1}{x-1}-\frac{1}{x+1}\,dx$$

$$=\ln|x-1|-\ln|x+1|\Big|_3^4$$

$$=\ln\left|\frac{x-1}{x+1}\right|\Big|_3^4=\ln\frac{3}{5}-\ln\frac{2}{4}$$

$$=\ln\frac{3/5}{1/2}=\boxed{\ln\frac{6}{5}}$$

(12) Use integration by parts to find $\int(\ln x)^2\,dx$

$u=(\ln x)^2 \qquad dv=dx$

$du=2\ln x\left(\frac{1}{x}\right)dx \qquad v=x$

$=\frac{2\ln x}{x}dx$

$$\int(\ln x)^2dx=x(\ln x)^2-\int x\left(\frac{2\ln x}{x}\right)dx$$

$$=x(\ln x)^2-2\int\ln x\,dx$$

$u=\ln x \quad dv=dx$

$du=\frac{1}{x}dx \quad v=x$

$$=x(\ln x)^2-2\left[x\ln x-\int x\left(\frac{1}{x}dx\right)\right]$$

$$=x(\ln x)^2-2x\ln x+2\int dx$$

$$=x(\ln x)^2-2x\ln x+2x+C$$

(13) Find $\int \sec^4 x \tan^4 x \, dx$

Since we have an even power of $\sec x$, we want to convert the rest (other than a $\sec^2 x$) to $\tan x$.

$$\int \sec^4 x \tan^4 x \, dx = \int \sec^2 x \sec^2 x \tan^4 x \, dx = \int \sec^2 x (1+\tan^2 x) \tan^4 x \, dx = \int \sec^2 x (\tan^4 x + \tan^6 x) \, dx$$

$u = \tan x \quad du = \sec^2 x \, dx$

$$= \int u^4 + u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C}$$

(14) Use trigonometric substitution to find $\int \frac{x^3}{\sqrt{1+x^2}} \, dx$

We use $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$

$$= \int \frac{\tan^3\theta}{\sqrt{1+\tan^2\theta}} \sec^2\theta \, d\theta = \int \frac{\tan^3\theta \sec^2\theta \, d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\tan^3\theta \sec^2\theta \, d\theta}{\sec\theta} = \int \tan^3\theta \sec\theta \, d\theta$$

Here, we want to pull out a $\sec\theta \tan\theta \, d\theta$ and convert the rest into $\sec\theta$.

$$= \int \tan^2\theta \sec\theta \tan\theta \, d\theta = \int (\sec^2\theta - 1) \sec\theta \tan\theta \, d\theta$$

$u = \sec\theta$
$du = \sec\theta \tan\theta \, d\theta$

$$= \int u^2 - 1 \, du = \frac{u^3}{3} - u + C = \frac{\sec^3\theta}{3} - \sec\theta + C$$

(15) Use partial fractions to find $\int \frac{3x+2}{x^3 - 4x^2 + 4x} \, dx$

$x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$

$$\frac{3x+2}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$3x+2 = A(x-2)^2 + B(x)(x-2) + Cx$

$3x+2 = Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx$

x^2: $0 = A + B$

x: $3 = -4A - 2B + C$

1: $2 = 4A$

$A = \frac{1}{2}$, $B = \frac{-1}{2}$, $C = 4$

(3rd equation) (plug A into 1st equation)

$$\int \frac{1}{2}\frac{1}{x} - \frac{1}{2}\frac{1}{x-2} + \frac{4}{(x-2)^2} \, dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x-2| - \frac{4}{x-2} + C$$

$$= \frac{1}{2}\ln\left|\frac{x}{x-2}\right| - \frac{4}{x-2} + C$$

(16) Find $\int \sin^2 x \, dx$

Method 1

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\frac{1}{2}\int 1-\cos 2x \, dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

$= \boxed{\frac{x}{2} - \frac{\sin 2x}{4} + C}$

Method 2

$\int \sin^2 x \, dx$

$u = \sin x \qquad dv = \sin x \, dx$

$du = \cos x \, dx \qquad v = -\cos x$

$\int \sin^2 x \, dx = -\sin x \cos x - \int -\cos^2 x \, dx$

$= -\sin x \cos x + \int \cos^2 x \, dx$

$= -\sin x \cos x + \int 1 - \sin^2 x \, dx$

$\int \sin^2 x \, dx = -\sin x \cos x + \int dx - \int \sin^2 x \, dx$

$2\int \sin^2 x \, dx = -\sin x \cos x + x$

$\int \sin^2 x \, dx = \boxed{\frac{x}{2} - \frac{\sin x \cos x}{2} + C}$

These are equal:

$\sin 2x = 2\sin x \cos x$

$\frac{\sin 2x}{2} = \sin x \cos x$

$\frac{x}{2} - \frac{\sin x \cos x}{2} + C$

$= \frac{x}{2} - \frac{\sin 2x}{4} + C$

(17) Find $\int \frac{x-1}{x^2-6x+5} \, dx$

$x^2 - 6x + 5 = (x-1)(x-5)$

$\frac{x-1}{(x-1)(x-5)} = \frac{1}{x-5}$

$\int \frac{1}{x-5} dx = \boxed{\ln|x-5| + C}$

If you do out the partial fractions:

$\frac{x-1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}$

$x - 1 = A(x-5) + B(x-1)$

x: $1 = A + B$

1: $-1 = -5A - B$

Adding: $0 = -4A$

$0 = A, \; B = 1$

$\int \frac{1}{x-5} dx = \ln|x-5| + C$