Midterm 2 Solutions

1. Find the area enclosed by the curves $x=y^2-5y$, $x=3y-y^2$

Intersection: $y^2-5y=3y-y^2$

$2y^2-8y=0$

$2y(y-4)=0$

$y=0,4$

@ $y=1$ $\quad (1)^2-5(1)=-4$

$3(1)-(1)^2=2$

So $x=3y-y^2$ is the larger of the two curves.

$$A=\int_0^4 (3y-y^2)-(y^2-5y)\,dy$$

$$=\int_0^4 8y-2y^2\,dy$$

$$=4y^2-\tfrac{2}{3}y^3\Big|_0^4$$

$$=4(4)^2-\tfrac{2}{3}(4)^3$$

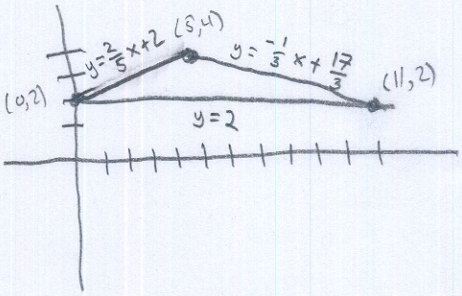
$$=4^3(1-\tfrac{2}{3})$$

$$=\boxed{\frac{64}{3}}$$

2. Use calculus to find the area of the triangle with vertices $(0,2)$, $(5,4)$, and $(11,2)$.

The line between $(0,2)$ and $(5,4)$ is $y=\frac{2}{5}x+2$

The line between $(5,4)$ and $(11,2)$ is $y=-\frac{1}{3}x+\frac{17}{3}$



This problem can be done either with respect to x or with respect to y:

With respect to x:

$$A=\int_0^5 \left(\tfrac{2}{5}x+2\right)-2\,dx+\int_5^{11}\left(-\tfrac{1}{3}x+\tfrac{17}{3}\right)-2\,dx$$

$$=\int_0^5 \tfrac{2}{5}x\,dx+\int_5^{11}-\tfrac{1}{3}x+\tfrac{11}{3}\,dx$$

$$=\tfrac{1}{5}x^2\Big|_0^5+\left(-\tfrac{1}{6}x^2+\tfrac{11}{3}x\right)\Big|_5^{11}$$

$$=\tfrac{1}{5}(5)^2+\left(-\tfrac{1}{6}(11)^2+\tfrac{11}{3}(11)\right)-\left(-\tfrac{1}{6}(5)^2+\tfrac{11}{3}(5)\right)$$

$$=5-\tfrac{121}{6}+\tfrac{121}{3}+\tfrac{25}{6}-\tfrac{55}{3}$$

$$=5-\tfrac{96}{6}+\tfrac{66}{3}$$

$$=5-16+22$$

$$=\boxed{11}$$

With respect to y:

$y=\frac{2}{5}x+2$

$y-2=\frac{2}{5}x$

$\frac{5}{2}y-5=\frac{5y-10}{2}=x$

$y=-\frac{1}{3}x+\frac{17}{3}$

$3y=17-x$

$x=17-3y$

$$A=\int_2^4 (17-3y)-\left(\tfrac{5}{2}y-5\right)dy$$

(The y-values range from 2 to 4)

$$=\int_2^4 22-\tfrac{11}{2}y\,dy$$

$$=\left[22y-\tfrac{11}{4}y^2\right]_2^4$$

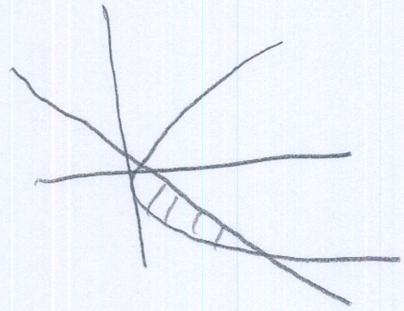
$$=\left(22(4)-\tfrac{11}{4}(4)^2\right)-\left(22(2)-\tfrac{11}{4}(2)^2\right)$$

$$=88-44-44+11$$

$$=\boxed{11}$$

Geometrically, the triangle has base 11, height 2, so $A=\frac{1}{2}bh=\frac{1}{2}(11)(2)=11$.

3. Use washers to find the volume of the solid obtained by rotating about the y-axis the region enclosed by $y^2 = x$ and $x = -2y$



Intersections:

$y^2 = -2y$

$y^2 + 2y = 0$

$y(y+2) = 0$

$y = -2, 0$

$r_i = y^2$

$r_o = -2y$

$V = \pi \int_{-2}^{0} (-2y)^2 - (y^2)^2 \, dy$

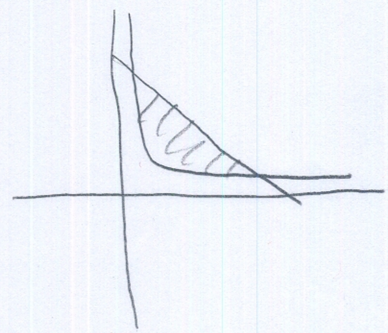
$= \pi \int_{-2}^{0} 4y^2 - y^4 \, dy$

$= \pi \left[\frac{4}{3} y^3 - \frac{y^5}{5} \right]_{-2}^{0}$

$= \pi \left[0 - \left(\frac{4}{3}(-2)^3 - \frac{(-2)^5}{5} \right) \right]$

$= \pi \left(\frac{32}{3} - \frac{32}{5} \right) = \pi \left(\frac{160}{15} - \frac{96}{15} \right) = \boxed{\frac{64\pi}{15}}$

4. Use washers to find the volume of the solid obtained by rotating about the x-axis the region enclosed by $y = 3/x$ and $y = 4 - x$



Intersections: $\frac{3}{x} = 4 - x$

$3 = 4x - x^2$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1, 3$

$r_o = 4 - x$

$r_i = \frac{3}{x}$

$V = \pi \int_{1}^{3} (4-x)^2 - \left(\frac{3}{x}\right)^2 dx$

$= \pi \int_{1}^{3} 16 - 8x + x^2 - \frac{9}{x^2} \, dx$

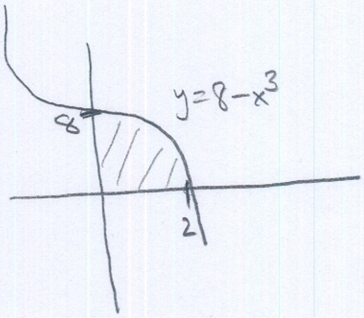
$= \pi \left[16x - 4x^2 + \frac{x^3}{3} + \frac{9}{x} \right]_{1}^{3}$

$= \pi \left([48 - 36 + 9 + 3] - [16 - 4 + \frac{1}{3} + 9] \right)$

$= \pi \left(24 - \frac{64}{3} \right) = \boxed{\frac{8\pi}{3}}$

5. Set up, but do not evaluate, the integral for the volume of the solid obtained by taking the region bounded by $y=8-x^3$, $x=0$, $y=0$ and rotating it about the y-axis using

a. disks or washers



This will be with respect to y, so we need to solve for x:

$y=8-x^3$
$x^3=8-y$
$x=\sqrt[3]{8-y}$

We will use disks here. The radius is our function:

$$V=\pi\int_0^8 \left(\sqrt[3]{8-y}\right)^2 dy$$
$$=\boxed{\pi\int_0^8 (8-y)^{2/3}dy}$$

b. Cylindrical shells

This will be with respect to x:

radius = x
height = $8-x^3$

$$\boxed{V=2\pi\int_0^2 x\left(8-x^3\right)dx}$$

6. Use cylindrical shells to find the volume obtained by rotating the region bounded by $y=\frac{1}{1+x^2}$, $y=0$, and $x=3$ about the y-axis.

Bounds: 0,3
radius: x
height: $\frac{1}{1+x^2}$

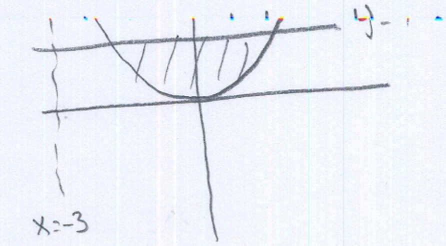
$$V=2\pi\int_0^3 x\,\frac{1}{1+x^2}\,dx$$
$$=\pi\int_0^3 \frac{2x}{1+x^2}dx$$

$u=1+x^2$
$du=2x\,dx$

New bounds: $0\to 1+(0)^2=1$
$3\to 1+(3)^2=10$

$$V=\pi\int_1^{10}\frac{1}{u}\,du$$
$$=\pi \ln u\Big|_1^{10}$$
$$=\pi\ln 10-\pi\ln 1$$
$$=\boxed{\pi\ln 10}$$

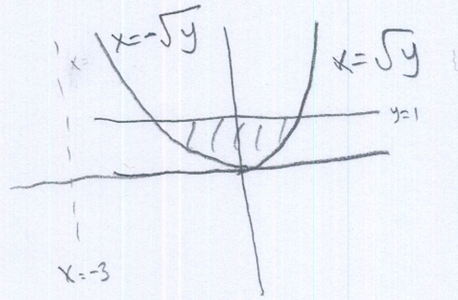
Discs/ Washers: use washers.



$y=x^2 \Rightarrow x=\pm\sqrt{y}$ (we need both to include the entire function)

$r_i = -\sqrt{y}-(-3) = 3-\sqrt{y}$

$r_o = \sqrt{y}-(-3) = 3+\sqrt{y}$



$V = \pi\int_0^1 (3+\sqrt{y})^2-(3-\sqrt{y})^2\,dy$

$= \pi\int_0^1 (9+6\sqrt{y}+y)-(9-6\sqrt{y}+y)\,dy$

$= \pi\int_0^1 12\sqrt{y}\,dy$

$= 12\pi\left(\frac{2}{3}y^{3/2}\right)\Big|_0^1$

$= 8\pi\, y^{3/2}\Big|_0^1 = \boxed{8\pi}$

Cylindrical: we integrate with respect to x.

radius: $x-(-3) = x+3$

height: $1-x^2$

bounds: $1=x^2$

$\pm 1 = x$

$V = 2\pi\int_{-1}^{1} (x+3)(1-x^2)\,dx$

$= 2\pi\int_{-1}^{1} x - x^3 + 3 - 3x^2\,dx$

$= 2\pi\left[\frac{x^2}{2}-\frac{x^4}{4}+3x-x^3\right]_{-1}^{1}$

$= 2\pi\left(\left[\frac{1}{2}-\frac{1}{4}+3-1\right]-\left[\frac{1}{2}-\frac{1}{4}-3+1\right]\right)$

$= 2\pi(3-1+3-1) = \boxed{8\pi}$

8. $m = 2\,kg$, $d = 6\,m$, $g = 10\,m/s^2$

The force acting on the pumpkin is its weight: $F = mg = (2)(10) = 20\,\frac{kg\,m}{s^2} = 20\,N$.

$$W = Fd = (20)(6) = 120\,\frac{kg\,m^2}{s^2} = \boxed{120\,J}$$

9. A spring has a natural length of 20 cm. If a 25N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

$F = kx$: $\quad 25 = k(.3 - .2) = .1k$ $\quad$ (we convert into meters here, so 20 cm = 0.2 m, etc.)

$k = 250\,N/m$

$$W = \int_{.2-.2}^{.25-.2} kx\,dx = \int_0^{.05} 250x\,dx = 125x^2\Big|_0^{.05} = 125(.05)^2 = \frac{125}{400} = \frac{5}{16}\,J$$

10. Find the average value of the function $h(x) = (3-2x)^{-2}$ on the interval $[-1,1]$

$$h_{ave} = \frac{1}{1-(-1)}\int_{-1}^{1}(3-2x)^{-2}\,dx$$

$u = 3-2x$

$du = -2\,dx$

$\frac{du}{-2} = dx$

Converting bounds, $-1 \to 3-2(-1) = 5$

$1 \to 3-2(1) = 1$

$$h_{ave} = \frac{1}{2}\int_5^1 u^{-2}\frac{du}{-2}$$

$$= \frac{-1}{4}\int_5^1 u^{-2}\,du$$

$$= \frac{1}{4}\int_1^5 u^{-2}\,du$$

$$= \frac{1}{4}\left[\frac{-1}{u}\right]_1^5 = \frac{1}{4}\left(\frac{-1}{5} - (-1)\right) = \frac{1}{4}\left(\frac{4}{5}\right) = \boxed{\frac{1}{5}}$$

11. Let $f(x) = (x-1)^3$ on $[1,5]$. Find a value c such that $f(c)$ is equal to the average value of f on the interval.

$$f_{ave} = \frac{1}{5-1}\int_1^5 (x-1)^3\,dx$$

$u = x-1$

$du = dx$

bounds: $1 \to 1-1 = 0$

$5 \to 5-1 = 4$

$$f_{ave} = \frac{1}{4}\int_0^4 u^3\,du$$

$$= \frac{1}{4}\left[\frac{u^4}{4}\right]_0^4$$

$$= \frac{1}{16}(4^4) = 16$$

$f(c) = 16$

$(c-1)^3 = 16$

$c - 1 = \sqrt[3]{16}$

$c = \boxed{1 + \sqrt[3]{16}}$

$= \boxed{1 + 2\sqrt[3]{2}}$ either

(either is fine)