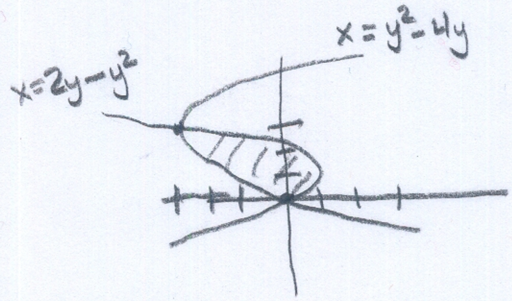
1. Find the area enclosed by the curves $x = y^2 - 4y$ and $x = 2y - y^2$



Intersections: $y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$2y(y-3) = 0$

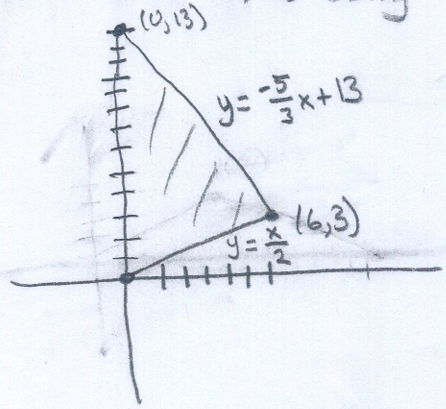
$y = 0, 3 \quad (0,0) \quad (-3,3)$

$$A = \int_0^3 (2y - y^2) - (y^2 - 4y)\, dy$$

$$= \int_0^3 6y - 2y^2\, dy$$

$$= \left[3y^2 - \frac{2y^3}{3}\right]_0^3 = 3(3)^2 - \frac{2(3)^3}{3} = 27 - 18 = \boxed{9}$$

2. Find the area of the triangle with vertices $(0,0)$, $(6,3)$ and $(0,13)$ using calculus.



$$A = \int_0^6 (13 - \tfrac{5}{3}x) - \frac{x}{2}\quad dx$$

$$= \int_0^6 13 - \frac{13x}{6}\, dx$$

$$= \left[13x - \frac{13x^2}{12}\right]_0^6 = 78 - \frac{13(36)}{12}$$

$$= 78 - 13(3)$$

$$= 78 - 39 = \boxed{39}$$

If we check geometrically, we have a triangle with base 13, height 6, so

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(13)(6) = 39.$$

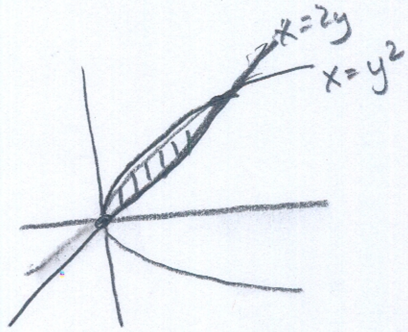
3. Use washers to find the volume of the solid obtained by rotating about the y-axis the region enclosed by $y^2 = x$, $x = 2y$.

Intersection: $y^2 = 2y$

$y^2 - 2y = 0$

$y(y-2) = 0$

$(0,0)$ $(4,2)$ are the intersection points.



$r_{outer} = 2y$

$r_{inner} = y^2$

$$V = \pi \int_0^2 r_o^2 - r_i^2 \, dy$$

$$= \pi \int_0^2 (2y)^2 - (y^2)^2 \, dy$$

$$= \pi \int_0^2 4y^2 - y^4 \, dy$$

$$= \pi \left[\frac{4y^3}{3} - \frac{y^5}{5}\right]_0^2$$

$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \boxed{\frac{64\pi}{15}}$$

4. Use washers to find the volume of the solid obtained by rotating about the x-axis the region enclosed by $y = \frac{1}{x}$ and $y = -\frac{1}{2}x + \frac{3}{2}$

Intersections: $\frac{1}{x} = \frac{3}{2} - \frac{1}{2}x$

$2 = 3x - x^2$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1, 2$

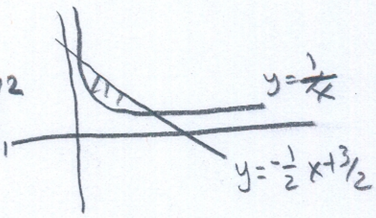
$r_o = 3/2 - \frac{x}{2}$

$r_i = \frac{1}{x}$

$$V = \pi \int_1^2 \left(\frac{3}{2} - \frac{x}{2}\right)^2 - \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^2 \frac{9}{4} - \frac{3x}{2} + \frac{x^2}{4} - \frac{1}{x^2} \, dx$$

$$= \pi\left[\frac{9}{4}x - \frac{3x^2}{4} + \frac{x^3}{12} + \frac{1}{x}\right]_1^2$$



$$= \pi\Big(\left[\frac{9}{4}(2) - \frac{3}{4}(2)^2 + \frac{2^3}{12} + \frac{1}{2}\right]$$

$$- \left[\frac{9}{4} - \frac{3}{4} + \frac{1}{12} + 1\right]\Big)$$

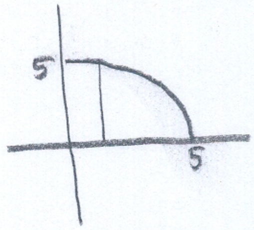
$$= \pi\left(\frac{9}{2} - 3 + \frac{2}{3} + \frac{1}{2} - \frac{9}{4} + \frac{3}{4} - \frac{1}{12} - 1\right)$$

$$\pi\left(\frac{54}{12} - \frac{36}{12} + \frac{8}{12} + \frac{6}{12} - \frac{27}{12} + \frac{9}{12} - \frac{1}{12} - \frac{12}{12}\right)$$

$$= \boxed{\frac{\pi}{12}}$$

5. Use disks to find the volume of the solid obtained by taking the region bounded by $y=\sqrt{25-x^2}$, $x=0$, and $y=0$ and

a. rotating it about the x-axis



$$V=\pi\int_0^5\left(\sqrt{25-x^2}\right)^2\,dx$$

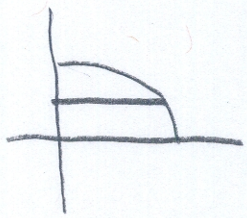
$$=\pi\int_0^5 25-x^2\,dx$$

$$=\pi\left[25x-\frac{x^3}{3}\right]_0^5$$

$$=\pi\left(125-\frac{125}{3}\right)$$

$$=\boxed{\frac{250\pi}{3}}$$

b. rotating it about the y-axis



$y=\sqrt{25-x^2}$

$y^2=25-x^2$

$x^2=25-y^2$

$x=\sqrt{25-y^2}$

$$V=\pi\int_0^5\left(\sqrt{25-y^2}\right)^2\,dy$$

$$=\pi\int_0^5 25-y^2\,dy$$

$$=\pi\left[25y-\frac{y^3}{3}\right]_0^5$$

$$=\pi\left(125-\frac{125}{3}\right)=\boxed{\frac{250\pi}{3}}$$

6. Use cylindrical shells to find the volume of the solid obtained by rotating about the y-axis the region bounded by $y=\sqrt{25-x^2}$, $y=0$, and $x=0$.

$$V=2\pi\int_0^5 x\sqrt{25-x^2}\,dx$$

$u=25-x^2$

$du=-2x\,dx \Rightarrow \frac{du}{-2}=x\,dx$

$$V=2\pi\int_{x=0}^{x=5}\sqrt{u}\,\frac{du}{-2}$$

$$=-\pi\int_{25}^{0}\sqrt{u}\,du$$

$$=\pi\int_0^{25}\sqrt{u}\,du$$

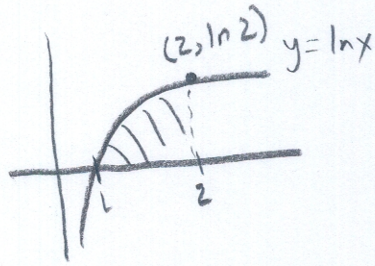
$$=\pi\left.\frac{u^{3/2}}{3/2}\right|_0^{25}$$

$$=\frac{2\pi}{3}\left.u^{3/2}\right|_0^{25}=\frac{2\pi}{3}(125-0)$$

$$=\boxed{\frac{250\pi}{3}}$$

7. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y=\ln x$, $y=0$, and $x=2$ about the y-axis.

Cylindrical: $\boxed{V=2\pi\int_1^2 x\ln x\,dx}$



Washer: bounds are 0, $\ln 2$

$r_o=2$

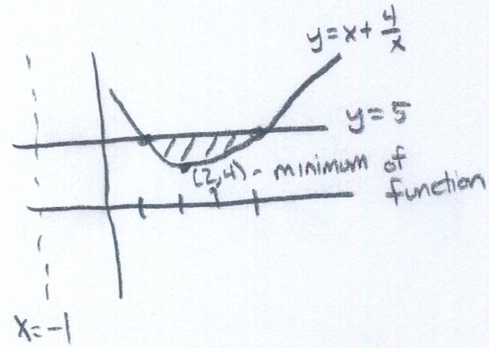
$r_i=e^y$

$$V=\pi\int_0^{\ln 2}(2^2-(e^y)^2)\,dy$$

$$=\boxed{\pi\int_0^{\ln 2}4-e^{2y}\,dy}$$

Note: you only need one of these. This is just to show it can be done either way.

8. Use any method to find the volume of the solid obtained by rotating the region bounded by $y=5$, $y=x+\frac{4}{x}$ about the line $x=-1$



Intersection: $5=x+\frac{4}{x}$

$5x=x^2+4$

$0=x^2-5x+4$

$0=(x-1)(x-4)$

$x=1,4$

Cylinders: radius $= x-(-1)=x+1$

height $=5-x-\frac{4}{x}$

$$V=2\pi\int_1^4(x+1)(5-x-\tfrac{4}{x})\,dx$$

$$=2\pi\int_1^4(5x-x^2-4+5-x-\tfrac{4}{x})\,dx$$

$$=2\pi\int_1^4(-x^2+4x+1-\tfrac{4}{x})\,dx$$

$$=2\pi\left[-\tfrac{x^3}{3}+2x^2+x-4\ln x\right]_1^4$$

$$=2\pi\left(\left[-\tfrac{64}{3}+32+4-4\ln 4\right]-\left[-\tfrac{1}{3}+2+1-4\ln 1\right]\right)$$

$$=2\pi\left(-\tfrac{63}{3}+33-4\ln 4\right)$$

$$=2\pi(12-4\ln 4)=\boxed{24\pi-8\pi\ln 4}$$

Washer: $y=x+\frac{4}{x}$

$\frac{dy}{dx}=1-\frac{4}{x^2}=0$

$x^2-4=0$

$x=\pm 2$

So the minimum is at $x=2$.

There are two values of x for a given y-value:

$x+\frac{4}{x}=c$

$x^2+4=cx$

$x^2-cx+4=0$

$x=\frac{c\pm\sqrt{c^2-16}}{2}$ by the Quadratic formula

bounds: $y=4$, $y=5$

$r_o: 1+\frac{y+\sqrt{y^2-16}}{2}$

$r_i: \frac{1+y-\sqrt{y^2-16}}{2}$

$$V=\pi\int_4^5\left(1+\tfrac{y+\sqrt{y^2-16}}{2}\right)^2-\left(1+\tfrac{y-\sqrt{y^2-16}}{2}\right)^2dy$$

$$=\pi\int_4^5\left(2\sqrt{y^2-16}+y\sqrt{y^2-16}\right)dy$$

Note: this method does not work well for this problem.

9. If the work required to stretch a spring one foot beyond its natural length is 12 ft-lbs, how much work is needed to stretch it 9 inches beyond its natural length? ⑤

$12 = \int_0^1 kx\,dx$ $\qquad$ $F = kx$ (Hooke's Law)

$12 = \frac{1}{2}kx^2\Big|_0^1 = \frac{1}{2}k$

$24 = k$ $\qquad$ ($k = 24\ \text{lb/ft}$)

9 inches $= \frac{3}{4}$ ft

$W = \int_0^{3/4} kx\,dx = \int_0^{3/4} 24x\,dx$

$= 12x^2\Big|_0^{3/4} = 12\left(\frac{3}{4}\right)^2 = \boxed{\frac{27}{4}\text{ ft-lbs.}}$

10. A spring has natural length 20 cm. Compare the work W_1 done in stretching the spring from 20 cm to 30 cm with the work W_2 done in stretching it from 30 cm to 40 cm. Find the relation between W_1 and W_2.

$W_1 = \int_0^{.1} kx\,dx = \frac{1}{2}kx^2\Big|_0^{.1} = \frac{k}{200}$ $\qquad$ $\boxed{W_2 = 3W_1}$

$W_2 = \int_{.1}^{.2} kx\,dx = \frac{1}{2}kx^2\Big|_{.1}^{.2} = \frac{k}{2}\left(\frac{4}{100} - \frac{1}{100}\right) = \frac{3k}{200}$

11. Find the average value of the function $h(x) = (3-2x)^{-1}$ on $[-1, 1]$

$f_{ave} = \frac{1}{1-(-1)}\int_{-1}^{1} \frac{1}{3-2x}\,dx$

$u = 3-2x$

$du = -2dx$

$\frac{du}{-2} = dx$

$f_{ave} = \frac{1}{2}\int_{x=-1}^{x=1} \frac{1}{u}\,\frac{du}{-2}$

$= \frac{-1}{4}\int_5^1 \frac{1}{u}\,du$

$= \frac{1}{4}\int_1^5 \frac{1}{u}\,du$

$= \frac{1}{4}\ln u\Big|_1^5$

$= \frac{1}{4}\ln 5 - \frac{1}{4}\ln 1$

$= \boxed{\frac{1}{4}\ln 5}$

12. Find the average value of $f(x)=\sqrt[3]{x}$ on $[1,8]$

$$=\frac{1}{7}\left.\frac{x^{4/3}}{4/3}\right|_1^8$$

$$=\frac{3}{28}\left(8^{4/3}-1^{4/3}\right)$$

$$=\frac{3}{28}(16-1)=\boxed{\frac{45}{28}}$$

13. Find the average value of $g(x)=\cos^4(x)\sin(x)$ on $[0,\pi]$.

$$g_{ave}=\frac{1}{\pi-0}\int_0^{\pi}\cos^4(x)\sin(x)$$

$u=\cos x$

$du=-\sin x\,dx$

$-du=\sin x\,dx$

$$g_{ave}=\frac{1}{\pi}\int_{x=0}^{x=\pi}u^4(-du)$$

$$=\frac{-1}{\pi}\int_1^{-1}u^4\,du$$

$$=\frac{1}{\pi}\int_{-1}^{1}u^4\,du$$

$$=\frac{1}{\pi}\left[\frac{u^5}{5}\right]_{-1}^{1}=\boxed{\frac{2}{5\pi}}$$

14. Let $f(x)=(x-3)^2$ on the interval $[2,5]$. Find a value c such that $f(c)$ is equal to the average value of f on the interval.

$$f_{ave}=\frac{1}{5-2}\int_2^5(x-3)^2\,dx$$

$u=x-3$

$du=dx$

$$=\frac{1}{3}\int_{x=2}^{x=5}u^2\,du$$

$$=\frac{1}{3}\int_{-1}^{2}u^2\,du$$

$$=\frac{1}{3}\left[\frac{u^3}{3}\right]_{-1}^{2}$$

$$=\frac{1}{9}\left[u^3\right]_{-1}^{2}$$

$$=\frac{1}{9}(8-(-1))=1$$

or: $f_{ave}=\frac{1}{3}\int_2^5(x-3)^2$

$$=\frac{1}{3}\int_2^5 x^2-6x+9\,dx$$

$$=\frac{1}{3}\left[\frac{x^3}{3}-3x^2+9x\right]_2^5$$

$$=\frac{1}{3}\left[\left(\frac{125}{3}-75+45\right)-\left(\frac{8}{3}-12+18\right)\right]$$

$$=\frac{1}{3}\left[\frac{117}{3}-30-6\right]$$

$$=\frac{1}{3}[39-36]$$

$$=1$$

$f_{ave}=f(c)$

$1=(c-3)^2$

$1=c^2-6c+9$

$0=c^2-6c+8$

$0=(c-2)(c-4)$

$\boxed{c=2\text{ or }4}$