(b) $x^2 y'' + xy' + (2x^2 - 1)y = 0$ around $x_0 = 0$.

Reguler singuler pt

$$p_0 = \lim_{x \to 0} x \frac{x}{x^2} = 1$$

$$q_0 = \lim_{x \to 0} \frac{x^2 \cdot (2x^2 - 1)}{x^2} = -1$$

So $F(r) = r(r-1) + r - 1 = r^2 - r + r - 1 = r^2 - 1$

$$\underbrace{r = \pm 1.}_{\text{roots differ by an integer.}}$$

For larger root,

$r = 1$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

<u>Plug in:</u> $x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$

$$+ (2x^2 - 1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} 2 a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Change indices to get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2}x^{n+r}$$

$$-\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$n=0$: $\underbrace{[r(r-1)+r-1]}_{\text{indicial equation}} a_0 = 0$

$n=1$: $[(r+1)r + (r+1) - 1]a_1 = 0 \Rightarrow a_1 = 0$

$n \geq 2$: $a_n = \dfrac{-2a_{n-2}}{(n+r)(n+r-1)+(n+r)-1}$

$$a_2 = \frac{-2a_0}{3\cdot 2+3-1} = -\frac{a_0}{4}$$

$a_3 = 0$

$$a_4 = \frac{-2a_2}{5\cdot 4+5-1} = -\frac{1}{12}a_2 = \frac{1}{48}a_0$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} = x - \frac{1}{4}x^3 + \frac{1}{48}x^5 - \cdots$$

$$y_2 = a\ln(x)y_1 + x^{-1} + \sum_{n=1}^{\infty} c_n x^{n-1}$$

$$y_2' = a\ln(x)y_1' + a\left(\frac{1}{x}\right)y_1 + \left(-\frac{1}{x^2}\right) + \sum_{n=2}^{\infty} c_n(n-1)x^{n-2}$$

$$y_2'' = a\ln(x)y_1'' + 2a\left(\frac{1}{x}\right)y_1' + a\left(-\frac{1}{x^2}\right)y_1 + \left(\frac{2}{x^3}\right) + \sum_{n=3}^{\infty}(n-1)(n-2)c_n x^{n-3}$$

Plug in:

$$2axy_1' - \cancel{ay_1} + \cancel{\frac{2}{x}} + \sum_{n=3}^{\infty}(n-1)(n-2)c_n x^{n-1} + \cancel{ay_1} - \cancel{\frac{1}{x}}$$

$$+ \sum_{n=2}^{\infty}(n-1)c_n x^{n-1} + 2x - \cancel{\frac{1}{x}} + \underbrace{2\sum_{n=1}^{\infty} c_n x^{n+1}}_{\text{change indices } n+1=m-1} - \sum_{n=1}^{\infty} c_n x^{n-1} = 0$$

We get:

$$2ax\left(1 - \frac{3}{4}x^2 + \frac{5}{48}x^4 + \cdots\right) + c_2 x + 2x - c_1 - c_2 x$$

$$+ \sum_{n=3}^{\infty}\left[\left((n-1)(n-2) + (n-1) - 1\right)c_n + 2c_{n-2}\right]x^{n-1} = 0$$

$$-c_1 = 0 \implies c_1 = 0$$

$$2ax + c_2 x + 2x - c_2 x = 0 \implies a = -1$$

n=3: $(2+2-1)c_3 + 2c_1 = 0 \implies c_3 = 0$

We can choose $c_2 = 0$ (arbitrary)

n=4: $-\frac{3}{4}(2a) + (6+3-1)c_4 + 2c_2 = 0$

$$\frac{3}{2} + 8c_4 = 0 \implies c_4 = -\frac{3}{16}$$

$n=5 \implies c_5 = 0$

$n=6$: $2a\left(\frac{5}{48}\right) + (20+5-1)c_6 + 2c_4 = 0$

$$-\frac{5}{24} - \frac{3}{8} + 24c_6 = 0 \implies c_6 = \frac{14}{24^2}.$$

So $y_2 = -\ln(x)\,y_1 + x^{-1}\left(1 - \frac{3}{16}x^4 + \frac{14}{24^2}x^6 + \dots\right)$