Homework 3 : Solutions

Students only get full credit if they show their work. Total of 60 pts.

3.3. 15) $y'' + y' + 1.25y = 0$.

Characteristic eqn:

$$r^2 + r + 1.25 = 0$$

2pts

$$r = \frac{-1 \pm \sqrt{1 - 4(1.25)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-4}}{2} = -\frac{1}{2} \pm i$$

So general solution is

$$y(t) = c_1 e^{-\frac{t}{2}} \cos t + c_2 e^{-\frac{t}{2}} \sin t$$

17) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$

3pts

Ch. eqn: $r^2 + 4 = 0$

$$r = \pm 2i$$

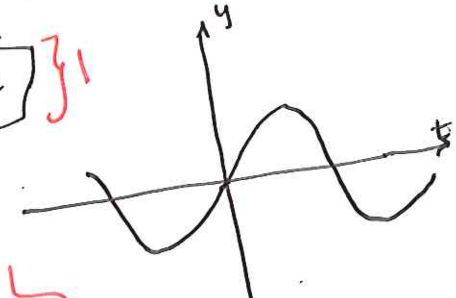
So gen soln: $y(t) = c_1 \cos 2t + c_2 \sin 2t$. $\}1$

$y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$. So $c_1 = 0$.

$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$

$y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 = 2c_2 = 1$, so $c_2 = \frac{1}{2}$.

Therefore, the solution is $\boxed{y(t) = \frac{1}{2} \sin 2t}$ $\}1$



Solution $\underbrace{\text{oscillates}}_{1}$ between $\frac{1}{2}$ & $-\frac{1}{2}$ steady.

19. $y'' - 2y' + 5y = 0$, $y(\frac{\pi}{2}) = 0, \; y'(\frac{\pi}{2}) = 2$

Ch. eqn: $r^2 - 2r + 5 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm 2i$$ 3pts

So general solution: $y(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t$ 1

$y(\frac{\pi}{2}) = c_1 e^{\pi/2} \cos(\pi) + c_2 e^{\pi/2} \sin(\pi) = -c_1 e^{\pi/2} = 0$

so $c_1 = 0$

$y'(t) = c_1 e^t \cos 2t + c_1 e^t \sin 2t \cancel{+} - 2c_1 e^t \sin 2t + 2c_2 e^t \cos 2t$

$y'(\frac{\pi}{2}) = c_1 e^{\pi/2} \cos(\pi) + c_2 e^{\pi/2} \sin(\pi) - 2c_1 e^{\pi/2} \sin(\pi) + 2c_2 e^{\pi/2} \cos(\pi)$

$= -c_1 e^{\pi/2} - 2c_2 e^{\pi/2} = 2$

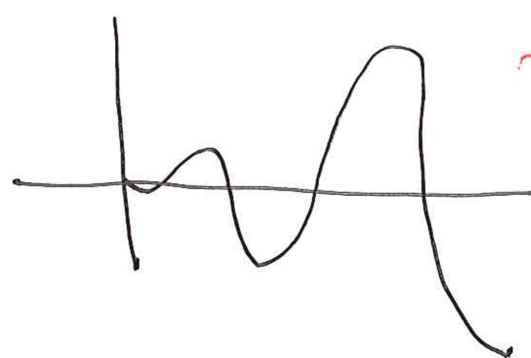
$-e^{\pi/2}(c_1 + 2c_2) = 2$ but $c_1 = 0$

So $-2e^{\pi/2} c_2 = 2$

$c_2 = -\frac{1}{e^{\pi/2}}$

so solution is: $y(t) = \cancel{\frac{1}{e}} - e^{-\pi/2 + t} \cancel{\sin} \sin 2t$ 1

as $t \to \infty$ behavior is oscillating growing in magnitude.



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23. $3u'' - u' + 2u = 0$, $u(0) = 2$, $u'(0) = 0$.

(a) Solution —

char eqn $3r^2 - r + 2 = 0$ part(a) — 2pts

roots $r = \dfrac{1 \pm \sqrt{1 - 4\cdot 2\cdot 3}}{6} = \dfrac{1}{6} \pm \dfrac{i\sqrt{23}}{6}$

So gen soln is $u(t) = c_1 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + \underbrace{c_2 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)}_{1}$

$u(0) = c_1 e^0 \cos 0 = c_1 = 2$

$u'(t) = \frac{1}{6}c_1 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + \frac{1}{6}c_2 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$

$- \frac{\sqrt{23}}{6}c_1 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + \frac{\sqrt{23}}{6}c_2 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right)$

$u'(0) = \frac{1}{6}c_1 + \frac{\sqrt{23}}{6}c_2 = 0$ and $c_1 = 2$

so $\frac{1}{3} + \frac{\sqrt{23}}{6}c_2 = 0$

$c_2 = -\frac{2}{\sqrt{23}}$

Therefore $u(t) = \overbrace{2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)}^{1}$

(b) for $t > 0$, find first time $|u(t)| > 10$.

(use computer) $t \approx 10.76$ part b 1pt

25. $y'' + 2y' + 6y = 0$, $y(0) = 2$, $y'(0) = \alpha \geq 0$

(a) find soln:

Char eqn: $r^2 + 2r + 6 = 0$

roots: $r = \dfrac{-2 \pm \sqrt{4 - 4(6)}}{2} = -1 \pm \sqrt{5}\, i$

gen soln: $y(t) = c_1 e^{-t} \cos(\sqrt{5}t) + c_2 e^{-t} \sin(\sqrt{5}t)$ } 1

$y(0) = c_1 e^0 \cos 0 = c_1 = 2.$

$y'(t) = -c_1 e^{-t} \cos(\sqrt{5}t) - c_2 e^{-t} \sin(\sqrt{5}t) - \sqrt{5} c_1 e^{-t} \sin(\sqrt{5}t)$
$+ \sqrt{5} c_2 e^{-t} \cos(\sqrt{5}t)$

$y'(0) = -c_1 + \sqrt{5} c_2 = \alpha$

$-2 + \sqrt{5} c_2 = \alpha$

part (a) 2pts

so $c_2 = \dfrac{\alpha + 2}{\sqrt{5}}$

$\boxed{\text{So } y(t) = 2e^{-t} \cos(\sqrt{5}t) + \dfrac{(\alpha+2)}{\sqrt{5}} e^{-t} \sin(\sqrt{5}t)}$ } 1

(b) Find α s.t. $y(1) = 0$

$y(1) = 2e^{-1} \cos\sqrt{5} + \dfrac{\alpha + 2}{\sqrt{5}} e^{-1} \sin\sqrt{5} = 0$

$\dfrac{\alpha+2}{\sqrt{5}} e^{-1} \sin\sqrt{5} = -2e^{-1} \cos\sqrt{5}$

$\alpha + 2 = \dfrac{-2\sqrt{5} \cos\sqrt{5}}{\sin\sqrt{5}}$

part b

1 pt

must show work.

$\alpha = -2\sqrt{5} \cot\sqrt{5} - 2 = 1.509$

(c) Find value of $t > 0$ for which $y = 0$.

Set $y(t) = 0$ and get

$$2e^{-t}\cos(\sqrt{5}\,t) + \left(\frac{q+2}{\sqrt{5}}\right)e^{-t}\sin(\sqrt{5}\,t) = 0$$

$e^{-t} > 0$, so cancel.

$$2\cos(\sqrt{5}\,t) + \left(\frac{q+2}{\sqrt{5}}\right)\sin(\sqrt{5}\,t) = 0.$$

We want to find t as a function of $q \geq 0$.

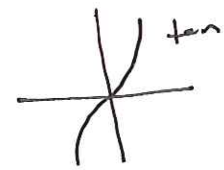
part (c) 2 pts

Rewrite this as $\tan(\sqrt{5}\,t) = -\dfrac{2\sqrt{5}}{q+2}$.

At this pt we may want to write

$\sqrt{5}\,t = \arctan\left(-\dfrac{2\sqrt{5}}{q+2}\right)$, but this is negative! (since $q \geq 0$)

We are looking for smallest <u>positive</u> value of t.



So, instead take $\sqrt{5}\,t = \pi - \arctan\left(+\dfrac{2\sqrt{5}}{q+2}\right)$,

which we can do since tan is an odd function.

So $$\boxed{t = \frac{1}{\sqrt{5}}\left(\pi - \arctan\left(\frac{2\sqrt{5}}{q+2}\right)\right)}$$

Also ok if students write $\sqrt{5}\,t = \pi + \arctan\left(-\dfrac{2\sqrt{5}}{q+2}\right)$ here

(d) $$\lim_{q\to\infty} \frac{1}{\sqrt{5}}\left(\pi - \arctan\left(\frac{2\sqrt{5}}{q+2}\right)\right) \longrightarrow \frac{1}{\sqrt{5}}\left(\pi - \arctan(0)\right) = \frac{\pi}{\sqrt{5}}.$$

part (d) 1 pt

3.4. 9. $25y'' - 20y' + 4y = 0$

Char eqn: $25r^2 - 20r + 4 = 0$

$(5r-2)(5r-2) = 0$ 1 pt

roots: $r = \frac{2}{5}$, repeated.

So gen soln: $\boxed{y(t) = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}}$

11. $9y'' - 12y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$

ch eqn: $9r^2 - 12r + 4 = 0$

$(3r-2)(3r-2) = 0$ 3pts

roots $r = \frac{2}{3}$, repeated

gen soln: $y(t) = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}$ } 1

$y(0) = c_1 = 2$

$y'(t) = \frac{2}{3} c_1 e^{\frac{2}{3}t} + c_2 e^{\frac{2}{3}t} + \frac{2}{3} c_2 t e^{\frac{2}{3}t}$

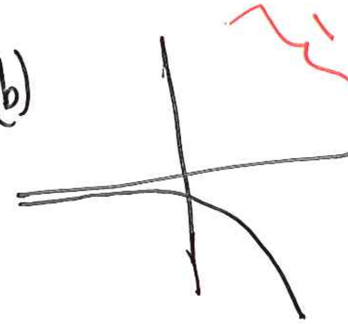
$y'(0) = \frac{2}{3} c_1 + c_2 = -1$

$\frac{4}{3} + c_2 = -1$

$c_2 = -\frac{7}{3}$ 1

So soln: $\boxed{y(t) = 2e^{2/3 t} - \frac{7}{3} t e^{\frac{2}{3}t}}$

(b) 1



So $y \to -\infty$ as $t \to \infty$

since $te^{2/3 t}$ goes negative faster than $e^{2/3 t}$ goes positive

13. $9y'' + 6y' + 82y = 0$, $y(0) = -1$, $y'(0) = 2$

ch eqn $= 9r^2 + 6r + 82 = 0$

roots: $r = \dfrac{-6 \pm \sqrt{36 - 4(9)(82)}}{2(9)} = \dfrac{-6}{18} \pm \dfrac{\sqrt{36(-81)}}{18}$

$= -\frac{1}{3} \pm 3i$

3pts

So gen soln: $y(t) = c_1 e^{-\frac{1}{3}t}\cos(3t) + c_2 e^{-\frac{1}{3}t}\sin(3t)$

$y(0) = c_1 = -1$

$y'(t) = -\frac{1}{3}c_1 e^{-\frac{1}{3}t}\cos 3t - \frac{1}{3}c_2 e^{-\frac{1}{3}t}\sin(3t) - 3c_1 e^{-\frac{1}{3}t}\sin 3t$
$+ 3c_2 e^{-\frac{1}{3}t}\cos(3t)$.

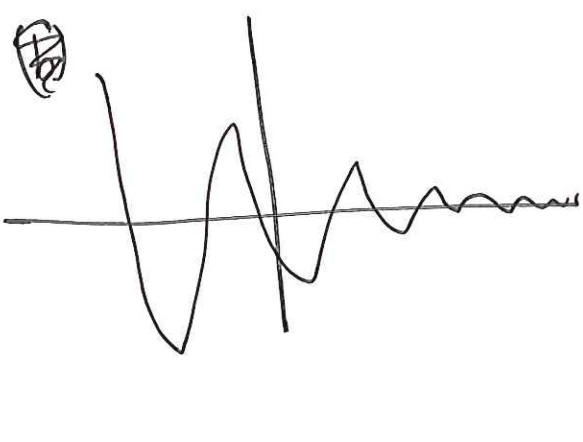
$y'(0) = -\frac{1}{3}c_1 + 3c_2 = 2$

$\frac{1}{3} + 3c_2 = 2$

$c_2 = \frac{5}{3}\cdot\frac{1}{3} = \frac{5}{9}$.

So soln is: $\boxed{y(t) = -e^{-\frac{1}{3}t}\cos 3t + \frac{5}{9}e^{-\frac{t}{3}}\sin 3t}$

(c)



So $y \to 0$ as $t \to \infty$

the exponential $e^{-\frac{t}{3}}$ forces everything to 0.

15. $4y'' + 12y' + 9y = 0$, $y(0) = 1$, $y'(0) = -4$.

(a) ch. eqn: $4r^2 + 12r + 9 = 0$

$(2r+3)(r+3) = 0$

$r = -\frac{3}{2}, -\frac{3}{2}$.

gen soln: $y(t) = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t}$

$y'(t) = -\frac{3}{2} c_1 e^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t} - \frac{3}{2} c_2 t e^{-\frac{3}{2}t}$

part(a) 3 pts

$y(0) = c_1 = 1$

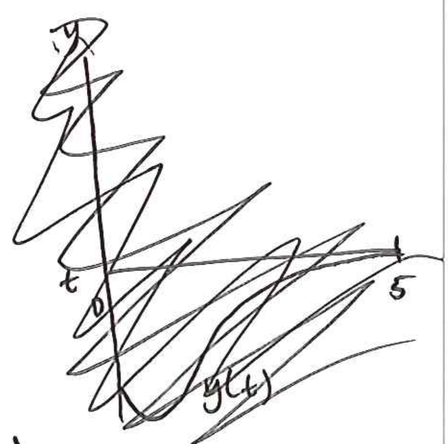
$y'(0) = -\frac{3}{2} c_1 + c_2 = -4$

$-\frac{3}{2} + c_2 = -4$

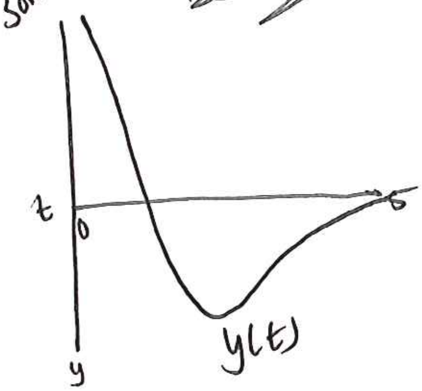
$c_2 = -4 + \frac{3}{2} = -\frac{5}{2}$

So solution is: $\boxed{y(t) = e^{-\frac{3}{2}t} - \frac{5}{2} t e^{-3/2 t}}$

~~graph:~~



plot of soln:



(b) $y(t) = 0$ when

$e^{-\frac{3}{2}t}\left(1 - \frac{5}{2}t\right) = 0$ part b 1pt

$1 = \frac{5}{2}t$, $t = \frac{2}{5}$

(c) when $y'(t) = 0$

$y'(t) = e^{-\frac{3}{2}t} \cdot \left(-\frac{3}{2}\right)\left(1 - \frac{5}{2}t\right) + \left(-\frac{5}{2}\right) e^{-\frac{3}{2}t} = 0$ part(c) 1pt

So when $\left(-\frac{3}{2} + \frac{15}{4}t - \frac{5}{2}\right) = 0$ ~~t = 15~~ $t = \frac{16}{15}$.

(d) Suppose now that $y'(0) = b$

We still have $c_1 = 1$

$$y'(0) = -\frac{3}{2}c_1 + c_2 = b$$

$$c_2 = b + \frac{3}{2}.$$

So solution is $y(t) = e^{-\frac{3}{2}t}\left(1 + \left(b+\frac{3}{2}\right)t\right)$ part(d) 1 pt

This intersects w/ $y=0$ when

$$1 + \left(b+\frac{3}{2}\right)t = 0$$

$t = -\dfrac{1}{b+\frac{3}{2}}$. If ~~b<~~ $b + \frac{3}{2} < 0$, then we have some t so that $y(t) = 0$.

So $b = -\frac{3}{2}$ is a critical point.

3.5 9. $u'' + \omega_0^2 u = \cos \omega t$, where $\omega_0^2 \neq \omega^2$.

3pts

Ch. eqn: $r^2 + \omega_0^2 = 0$

$$r = \pm i\omega_0$$

homog. soln: $y_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$

Guess part soln: $Y_p(t) = A\cos\omega t + B\sin\omega t.$

$$Y_p'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$Y_p''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t.$$

Plug in:

$$-A\omega^2\cos\omega t - B\omega^2\sin\omega t + A\omega_0^2\cos\omega t + B\omega_0^2\sin\omega t = \cos\omega t.$$

So $A\omega_0^2 - A\omega^2 = 1 \qquad A = \frac{1}{\omega_0^2-\omega^2}$

$B\omega_0^2 - B\omega^2 = 0 \qquad B = 0$

So $Y_p(t) = \frac{1}{\omega_0^2-\omega^2}\cos\omega t$

Gen Solution : $\boxed{y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{1}{\omega_0^2-\omega^2}\cos(\omega t)}$

11. $y'' + y' + 4y = 2\sinh t$ (Hint: $\sinh t = \frac{e^t - e^{-t}}{2}$)

Ch eqn: $r^2 + r + 4 = 0$

$$r = \frac{-1 \pm \sqrt{1-4(4)}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{15}\, i$$

3pts

homog soln: $y_h(t) = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)$

~~Guess. ...~~ Want to solve

$y'' + y' + 4y = e^t - e^{-t}$. 2 separate problems.

$Y_{p_1}(t) = Ae^t$, $Y_{p_2}(t) = Be^{-t}$

$Y_{p_1}' = Ae^t$ $\qquad Y_{p_2}'(t) = -Be^{-t}$

$Y_{p_1}'' = Ae^t$ $\qquad Y_{p_2}'' = Be^{-t}$

Plugging in,

$Ae^t + Ae^t + 4Ae^t = e^t$

$6A = 1$

$A = \frac{1}{6}$

$Be^{-t} - Be^{-t} + 4Be^{-t} = -e^{-t}$

$4B = -1$

$B = -\frac{1}{4}$

So solution is

$$\boxed{y(t) = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{15}}{2} t\right) + c_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{15}}{2} t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}}$$

13. $y'' + y' - 2y = 2t$, $y(0) = 0$, $y'(0) = 1$

ch eqn: $r^2 + r - 2 = 0$

$(r+2)(r-1)$

roots: $r = -2, 1$

homog. soln: $y_h(t) = c_1 e^{-2t} + c_2 e^t$

3pts

Guess $Y_p(t) = At + B$

$Y_p' = A$

$Y_p'' = 0$

Plug in: $A - 2(At + B) = 2t$

$\underbrace{-2A}_{2}t + (\underbrace{A - 2B}_{0}) = 2t + 0$

So $A = -1$ $B = -\frac{1}{2}$.

So gen soln $y(t) = c_1 e^{-2t} + c_2 e^{t} - t - \frac{1}{2}$

$y(0) = c_1 + c_2 - \frac{1}{2} = 0$

$c_1 + c_2 = \frac{1}{2}$

$y'(t) = -2c_1 e^{-2t} + c_2 e^{t} - 1$

$y'(0) = -2c_1 + c_2 - 1 = 1$

$-2c_1 + c_2 = 2$

Solving $c_2 = \cancel{[illegible]}\ 1$

$c_1 = \cancel{[illegible]} - \frac{1}{2}$

so we get $\boxed{y(t) = e^{-2t} - \frac{1}{2}e^{t} - t - \frac{1}{2}}$

15. $y'' - 2y' + y = te^{t} + 4$, $y(0) = 1$, $y'(0) = 1$

ch eqn: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

4 pts

$r = 1, 1.$

homog. ~~gen~~ soln: $y_h(t) = c_1 e^{t} + c_2 t e^{t}$ } 1

Solve two diff. for $g_1(t) = te^{t}$ and $g_2(t) = 4$.

Guess: $Y_{p_1}(t) = t^2 \cdot (At + B) e^{t}$ and $Y_{p_2}(t) = C$

$Y_{p_1}' = (At^3 + Bt^2)e^{t} + (3At^2 + 2Bt)e^{t}$ choosing right Y_p $Y_{p_2}' = Y_{p_2}'' = 0$

$$Y_{p_1}' = \left(At^3 + (3A+B)t^2 + 2Bt\right)e^t$$

$$Y_{p_1}'' = \left(At^3 + (3A+B)t^2 + 2Bt\right)e^t + \left(3At^2 + 2(3A+B)t + 2B\right)e^t$$

$$= \left(At^3 + (6A+B)t^2 + (6A+4B)t + 2B\right)e^t$$

Plugging in,

$$e^t\left(At^3 + (6A+B)t^2 + (6A+4B)t + 2B - 2\left(At^3 + (3A+B)t^2 + 2Bt\right) + \left(At^3 + Bt^2\right)\right)$$
$$= te^t$$

So t^3: $A - 2A + A = 0$ ✓

t^2: $6A + B - 2(\cancel{A}3A + B) + B = 0$ ✓

t: $6A + 4B + 2B = 1$

1: $2B = 0$

$B = 0$

$6A = 1$

$A = \frac{1}{6}$

$$\left.\begin{aligned} \text{so } Y_{p_1} &= t^2\left(\tfrac{1}{6}t\right)e^t \\ &= \tfrac{1}{6}t^3e^t \end{aligned}\right\} 1$$

~~So gen solution~~

for Y_{p_2}: $0 - 2\cdot 0 + C = 4$

$C = 4$

So $Y_{p_2} = 4$.

Therefore the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + \frac{1}{6} t^3 e^t + 4$$

IVP: $y(0) = 1$, $y'(0) = 1$

$$y(0) = c_1 + 4 = 1$$

$$c_1 = -3$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + \frac{1}{2} t^2 e^t + \frac{1}{6} t^3 e^t,$$

$$y'(0) = c_1 + c_2 = +1$$

$$-3 + c_2 = +1$$

$$c_2 = 4$$

So $\boxed{y(t) = -3e^t + 4 t e^t + \frac{1}{6} t^3 e^t + 4}$ } 1

31. If $Y_1(t)$ & $Y_2(t)$ are solns of $ay'' + by' + cy = g(t)$, $a, b, c > 0$.
Show $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$.

4 pts

1 for each case

① Case 1: real distinct, so $b^2 - 4ac > 0$. Then roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = r_1, r_2$$

- $Y_1 - Y_2$ is homog. solution, so of form
$c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Since $-\frac{b}{2a}$ is negative

and $\sqrt{b^2-4ac} < \sqrt{b^2} = b$,

then both roots are negative.

so $c_1 e^{r_1 t} + c_2 e^{r_2 t} \longrightarrow 0$ as $t \to 0$

since $e^{rt} \to 0$ if $r < 0$.

② Case 2: Complex. So $b^2 - 4ac < 0$

then roots : $r = \underbrace{\frac{-b}{2a}}_{\text{negative}} \pm i \cdot \underbrace{\frac{\sqrt{4ac-b^2}}{2a}}_{\text{positive}}$

So we get $Y_1 - Y_2 = c_1 e^{-\frac{b}{2a}t} \underbrace{\cos\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)}_{\text{btwn } 0\&1} + c_2 e^{-\frac{b}{2a}t} \underbrace{\sin\left(\frac{\sqrt{4ac-b^2}}{2a} t\right)}_{\text{btwn } 0\&1}$

and $e^{rt} \longrightarrow 0$ as $t \to \infty$
if $r < 0$

$-\frac{b}{2a}$ is neg, so $Y_1 - Y_2 \longrightarrow 0$ as $t \to \infty$.

③ Case 3: repeated roots., So $b^2 - 4ac = 0$

$r = -\frac{b}{2a}$, negative!

$Y_1 - Y_2 = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}$.

and therefore,

$$\lim_{t\to\infty}\left(C_1 e^{-\frac{b}{2a}t} + C_2 t e^{-\frac{b}{2a}t}\right) = 0 + 0 = 0$$

Since exp goes to 0 faster than $t \to \infty$.

(4) Case (4) If ~~R~~ $b=0$, then

$$ay'' + cy = g(t)$$

and roots of ch. eqn is $\pm i \frac{\sqrt{4ac}}{2a}$,

so we get $Y_1 - Y_2 = C_1 \cos\left(\frac{\sqrt{4ac}}{2a}\right) + C_2 \sin\left(\frac{\sqrt{4ac}}{2a}\right)$, which oscillates and does not go to 0 as $t \to \infty$.

3.6 7. $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$.

homog soln: $r^2 + 4r + 4 = 0$

$(r+2)(r+2)$

$r = -2, -2.$

4 pts

$$y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

$Y_p = V_1 \cdot e^{-2t} + V_2 \cdot te^{-2t}$. Let's find v_1, v_2.

$$V_1' = -\frac{te^{-2t} \cdot t^{-2}e^{-2t}}{W(e^{-2t}, te^{-2t})}, \quad V_2' = \frac{e^{-2t}\, t^{-2}e^{-2t}}{W(e^{-2t}, te^{-2t})}.$$

$$W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t}+e^{-2t} \end{vmatrix}$$

$$= e^{-2t}(-2te^{-2t}+e^{-2t}) + 2e^{-2t}(te^{-2t})$$

$$= -2te^{-4t} + e^{-4t} + 2te^{-4t} = e^{-4t} \Big\} 1$$

So $v_1' = \dfrac{-te^{-2t} \cdot t^{-2}e^{-2t}}{e^{-4t}} = -\dfrac{1}{t}$ so $v_1 = -\ln|t| \Big\} 1$

$v_2' = \dfrac{e^{-2t} \cdot t^{-2} \cdot e^{-2t}}{e^{-4t}} = \dfrac{1}{t^2}$ $v_2 = -\dfrac{1}{t} \Big\} 1$

So $Y_p(t) = -e^{-2t}\ln|t| - \underbrace{\frac{t}{t}e^{-2t}}_{\text{absorbed by } c_1e^{-2t}}$.

So $\boxed{y(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}(\ln(t))} \Big\} 1$

don't need abs value since $t>0$.

10.) $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$

r^2-2r+1, $r=1$

hom. soln: $y_h(t) = c_1e^t + c_2te^t$

4 pts

$Y_p = v_1e^t + v_2te^t$

$$V_1' = \frac{-te^{t} \cdot e^t/(1+t^2)}{W(e^{t}, te^{t})} \qquad V_2' = \frac{\cancel{te}\ e^{t} \cdot e^t/(1+t^2)}{W(e^{t}, te^{t})}$$

$$W(e^{-t}, te^{-t}) = \begin{vmatrix} e^{t} & te^{t} \\ e^{t} & te^{t} + e^{t} \end{vmatrix}$$

$$= e^t(te^t + e^t) - e^t(te^t)$$

$$= e^{2t} \quad \}1$$

$$V_1' = \frac{-t}{1+t^2} \qquad V_2' = \frac{1}{1+t^2}$$

$$V_1 = -\frac{1}{2}\ln|1+t^2| \;\}1 \qquad V_2 = \arctan(t) \;\}1$$

So $Y_p = -\frac{1}{2}\ln|1+t^2|e^t + te^t \arctan t$

1

~~Sy~~ Soln: $\boxed{y(t) = C_1 e^{t} + C_2 te^t - \frac{1}{2}\ln|1+t^2|e^t + te^t \arctan t}$

17. $x^2y'' - 3xy' + 4y = x^2 \ln x$, $x>0$, $y_1(x) = x^2$, $y_2(x) = x^2 \ln x$

5pts

① Verify: $y_1' = 2x$, $y_1'' = 2$ ← 1pt → ② $y_2' = 2x\ln x + x$

$y_2'' = 2\ln x + 3$

Plug in: $2x^2 - 3x(2x) + 4(x^2) = 0$ ✓

Plug in: $(2\ln x + 3)x^2 - 3x(2x\ln x + x) + 4x^2\ln x = 0$ ✓

③ Find particular solution

<u>homog</u>: $c_1 x^2 + c_2 x^2 \ln x$

$$y_p = v_1 x^2 + v_2 x^2 \ln x \quad , \quad g(x) = \ln(x)$$

$$v_1' = \frac{x^2 \ln x \cdot \ln(x)}{W(x^2, x^2 \ln(x))} \qquad v_2' = \frac{x^2 \cdot \ln x}{W(x^2, x^2 \ln(x))}$$

$$W(x^2, x^2 \ln(x)) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = 2x^3 \ln x + x^3 - 2x^3 \ln x$$
$$= x^3$$

$$v_1' = \frac{(\ln(x))^2}{x} \qquad v_2' = \frac{\ln x}{x}$$

$$v_1 = -\frac{1}{3}(\ln x)^3 \qquad v_2 = \frac{1}{2}(\ln x)^2$$

$$y_p = -\frac{1}{3}x^2(\ln x)^3 + \frac{1}{2}x^2(\ln x)^3 = \boxed{\frac{1}{6}x^2(\ln x)^3}$$