Today, we did the example

$$x y'' + y = 0.$$

$x=0$ is a regular singular point.

In class, we found that one solution was $y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots$

What about the second solution?

Since $r_1 - r_2 = 1 - 0 = 1$ is a positive integer, we know

y_2 will have the form

$$y_2(x) = a y_1 \ln(x) + x^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right]$$

since $r_2 = 0$, this is

$$y_2(x) = a y_1 \ln(x) + \left[1 + \sum_{n=1}^{\infty} c_n x^n\right]$$

Our goal is to find a and $c_1, c_2, \dots$

Let's plug in.

$$y_2'(x) = a y_1' \ln(x) + a y_1 \frac{1}{x} + \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y_2''(x) = a y_1'' \ln(x) + a y_1' \frac{1}{x} + a y_1' \frac{1}{x} + a y_1\left(-\frac{1}{x^2}\right) + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Let's plug into formula $x^2 y'' + x y = 0$.

<u>We get:</u>

$$a x^2 y_1'' \ln(x) + 2 a x y_1' - a y_1 + \sum_{n=2}^{\infty} n(n-1) c_n x^n + a x y_1 \ln(x)$$

$$+ x + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Notice that all log-terms disappear. This follow from the fact that y_1 is a solution to $x^2y'' + xy = 0$.

Therefore, we get:

$$(*) \quad 2axy_1' - ay_1 + x + \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} c_{n-1} x^n = 0$$

We want to find a and c_n for $n = 1, 2, 3, \ldots$

Notice that since $y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$

then $y_1' = 1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots$

So our equation (*) above can be rewritten as

$$2a\left(x - x^2 + \frac{1}{4}x^3 - \frac{1}{36}x^4 + \cdots\right) - a\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right)$$
$$+ x + (2c_2 + c_1)x^2 + (6c_3 + c_2)x^3 + (12c_4 + c_3) + \cdots = 0$$

Look at coeff of x^n for $n = 1, 2, \ldots$ and set $= 0$.

$\underline{n=1,\text{ coef of } x}$: $2a - a + 1 = 0 \Longrightarrow a = -1$.

$c_N = c_1$ is arbitrary so take it to be $= 0$.

$\underline{n=2:\text{ coeff of } x^2}$: $-2a + \frac{a}{2} + (2c_2 + c_1) = 0$

$$\Downarrow$$

$$2 - \frac{1}{2} + 2c_2 = 0 \Rightarrow c_2 = -\frac{3}{4}$$

$\underline{n=3,\ \text{coeff of } x^3}$: $\frac{2a}{4} - \frac{9}{12} + (6c_3 + c_2) = 0$

$$\downarrow$$

$$-\frac{1}{2} + \frac{1}{12} + 6c_3 + \left(-\frac{3}{4}\right) = 0$$

$$6c_3 = \frac{3}{4} + \frac{1}{2} - \frac{1}{12} = \frac{14}{12} = \frac{7}{6}$$

$$c_3 = \frac{7}{36}$$

and so on...

so $\quad y_2 = a\, y_1 \ln(x) + \left[1 + c_1 x + c_2 x^2 + \dots\right]$

and thus

$$\boxed{y_2 = -y_1 \ln(x) + 1 + \left(-\frac{3}{4}\right)x^2 + \frac{7}{36}x^3 + \dots}$$

The general solution is $\quad y(x) + c_1 y_1 + c_2 y_2$.