During Class, we found one solution to

$$2x^2y'' + 3xy' + (2x^2-1)y = 0$$ around regular singular pt $x_0 = 0$.

First, we found indicial equation

$$2r^2 + r - 1 = 0$$
$$(2r-1)(r+1) = 0$$

w/ roots $\quad r = \frac{1}{2}, -1$

The solution $y_1(x)$ associated to $r = \frac{1}{2}$ is:

$$y_1(x) = x^{1/2}\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k!\, 7 \cdot 11 \cdot \ldots \cdot (4k+3)}\right)$$

What about the solution associated to $r=-1$?

The general recurrence relation is:

$n=1$: $\quad a_1\left(2(r+1)r + 3(r+1) - 1\right) = 0 \qquad$ (coeff of x^{r+1})

$n \geq 2$: $\quad a_n = \dfrac{-2a_{n-2}}{(2n+2r-1)(n+r+1)} \qquad$ (coeff of x^{n+r})

and thus, for $r=-1$

$$a_n = \frac{-2a_{n-2}}{n(2n-3)}$$

Let's see a few terms.

$n=1$: $a_1(-1)=0 \Rightarrow a_1=0$.

$n=2$: $a_2 = \dfrac{-2a_0}{2(4-3)} = \dfrac{-a_0}{1}$

$n=3$ $a_3 = \dfrac{-2a_1}{3(6-3)} = 0$ since $a_1=0$.

$\therefore$ In fact, $a_{2k+1}=0$ for all k.

$n=4$: $a_4 = \dfrac{-2a_2}{4(8-3)} = \dfrac{-a_2}{2\cdot 5} = \dfrac{(-1)^2 a_0}{2\cdot 1\cdot 5}$

$n=6$: $a_6 = \dfrac{-2a_4}{6(12-3)} = \dfrac{-a_4}{3\cdot 9} = \dfrac{(-1)^3 a_0}{3\cdot 2\cdot 1\cdot 5\cdot 9}$

$n=8$: $a_8 = \dfrac{-2a_6}{8(16-3)} = \dfrac{-a_6}{4(13)} = \dfrac{(-1)^4 a_0}{4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 9\cdot 13}$

$\vdots$

$n=2k$ (even)

$a_{2k} = \dfrac{(-1)^k a_0}{k!\, 5\cdot 9\cdot \dots \cdot (4k-3)}$

Second solution? $y_2(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^{n+r}$ where $r=-1$, and a_n is defined above, 0 when n is odd and a_{2k} when $n=2k$, even.

So second solution is

$$y_2(x) = \left(a_0 x^{-1} + \sum_{k=1}^{\infty} \frac{(-1)^k a_0 \, x^{2k-1}}{k!\, 5 \cdot 9 \cdot \cdots (4k-3)} \right)$$

But a_0 can be "absorbed" by c_2. (it is a constant)

and we can pull out a common factor of x^{-1}.

$$y_2 = x^{-1} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k!\, 5 \cdot 9 \cdot \cdots (4k-3)} \right).$$

So the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\boxed{y(x) = c_1 x^{1/2} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k!\, 7 \cdots (4k+3)} \right) + c_2 x^{-1} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{k!\, 5 \cdots (4k-3)} \right)}$$