Review Session for Math 23 Midterm: SOLUTIONS

Chapter 1:

1. State the order and whether the following are linear/non-linear:

   (a) $y^{(4)}(t+\cos(t)) = e^t y'$ **Solution:** Linear, 4th order

   (b) $\csc(t)y^2 + \frac{1}{t}y' = 0$ **Solution:** Nonlinear, 1st order

2. For which values of r is t^r a solution to $t^2y'' - 4ty' + 4y = 0$ for $t > 0$?

   Solution: If $y(t) = t^r$, then $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Plugging in, we find:

   $$t^2(r(r-1)t^{r-2}) - 4t(rt^{r-1}) + 4t^r = 0$$

   which reduces to

   $$r^2 - 5r + 4 = 0.$$

   Therefore, $r = 4$ or $r = 1$.

Chapter 2:

3. Find the solution to each:

   (a) $t^2y' + ty = t^2e^t$

   Solution: First, get it in the right form:

   $$y' + \frac{1}{t}y = e^t.$$

   Next, we find the integrating factor:

   $$\mu(t) = \exp\int \frac{1}{t}\,dt = e^{\ln(t)} = t.$$

   Multiplying by the integrating factor, we get

   $$(ty)' = te^t.$$

   After integrating both sides (using IBP), we find that

   $$ty = te^t - e^t$$

   and thus $y = e^t - e^t/t$.

   (b) $y' = \dfrac{e^{2t}}{y}$

   Solution: This is separable, so we obtain:

   $$y\,dy = e^{2t}\,dt.$$

   Integrating both sides, we get

   $$y^2/2 = e^{2t}/2 + c.$$

   Therefore we implicitly get the solution $y^2 = e^{2t} + c$.

(c) $(2xy^2) + (2x^2y + 2y)y' = 0$

Solution: We can check that this equation is exact:

$$M_y = 4xy = N_x.$$

We want to find a solution of the form $\Phi(x, y) = c$ where $\Phi_x = M$ and $\Phi_y = N$. Integrating M with respect to x gets:

$$\Phi = x^2y^2 + h(y).$$

Now taking Φ_y and comparing to N to find h:

$$\Phi_y = 2x^2y + h'(y)$$

implies that $h'(y) = 2y$ and so $h(y) = y^2$. Thus, the solution is

$$x^2y^2 + y^2 = c.$$

(d) $1 + (x/y - \sin(y))y' = 0$

Solution: This is not exact since

$$M_y = 0 \neq 1/y = N_x.$$

However, we can find an integrating factor $\mu(y)$ since

$$\frac{N_x - M_y}{M} = 1/y$$

is a function of y only. Solving

$$\mu' = \mu/y,$$

we find that $\mu = y$. So solving the original problem is equivalent to solving the exact equation: $y + (x - y\sin(y))y' = 0$. We solve as above to find that the solution is:

$$xy + y\cos(y) - \sin(y) = c.$$

4. For which t and y do the following have a unique solution?

(a) $t^2y' + 3ty = \cos(t)$

Solution: This is a linear DE. Putting in the right form,

$$y' = \frac{3}{t}y = \frac{\cos(t)}{t^2}.$$

Since $p(t)$ and $g(t)$ are discontinuous at 0, a unique solution may not exist when $t = 0$.

(b) $y' = \dfrac{3t}{3y - y^2}$

Solution: This is nonlinear, so we should use Picard's theorem. Checking where $f(t, y)$ and $f_y(t, y)$ are discontinuous, we see a unique solution may not exist when $y = 0$ or $y = 3$.

Chapter 3:

5. Find the solution to each:

(a) $3y'' + 5y' + 2y = 0$

Solution: The characteristic polynomial is $3r^2 + 5r + 2 = 0$ which has roots $r = -1, -2/3$. Therefore the solution is:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t/3}.$$

(b) $4y'' + 9y = \cos 2t$

Solution: The characteristic polynomial is $4r^2 + 9 = 0$ which has roots $r = \pm 3i/2$. Therefore the homogeneous solution is:

$$y_h(t) = c_1 \cos(3t/2) + c_2 \sin(3t/2).$$

We can use the method of undetermined coefficients. We guess a particular solution

$$Y_P(t) = A\cos(2t) + B\sin(2t).$$

This has derivatives

$$Y_P'(t) = -2A\sin(2t) + 2B\cos(2t)$$

and

$$Y_P''(t) = -4A\cos(2t) - 4B\sin(2t).$$

Plugging in, we find that

$$4(-4A\cos(2t) - 4B\sin(2t)) + 9(A\cos(2t) + B\sin(2t)) = \cos(2t)$$

and thus $-16A + 9A = 1$ and $-16B + 9B = 0$, which implies that $A = -1/7$ and $B = 0$. Therefore the general solution is

$$y(t) = c_1 \cos(3t/2) + c_2 \sin(3t/2) - \frac{1}{7}\cos(2t).$$

(c) $y'' - 6y' + 9y = e^{3t}/t$

Solution: The characteristic polynomial is $r^2 - 6r + 9 = 0$, which has roots $r = 3, 3$. Therefore the homogeneous solution is

$$y_h(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

We should use variation of parameters to find a solution of the form $Y_P(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$. Let's find the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & 3te^{3t} + e^{3t} \end{vmatrix} = -e^{6t}.$$

To find v_1 and v_2, we solve certain 1st order differential equations.

$$v_1' = \frac{-te^{3t} \cdot e^{3t}/t}{-e^{6t}} = 1$$

and

$$v_2' = \frac{e^{3t} \cdot e^{3t}/t}{-e^{6t}} = -1/t$$

and so we find that $v_1 = t$ and $v_2 = -\ln(t)$. Thus, we find that the general solution is: $y(t) = c_1 e^{3t} + c_2 te^{3t} - \ln(t)te^{3t}$.

6. Given that $y(t) = e^t$ is one solution of $(t-1)y'' - ty' + y = 0$, for $t > 1$, find a second solution using reduction of order.

   Solution: Suppose $y(t) = v(t)e^t$ is a solution. Then $y'(t) = v'e^t + ve^t$ and $y''(t) = v''e_t + 2v'e^t + ve^t$. Plugging in, we get that

$$(t-1)(v''e_t + 2v'e^t + ve^t) - t(v'e^t + ve^t) + ve^t = 0.$$

   We get some cancellation, and find that $(t-1)e^t v'' + (t-2)e^t v' = 0$. Let $w = v'$ and we obtain a first order DE:

$$w' + \frac{(t-2)}{(t-1)}w = 0.$$

   Solving this first order DE, we get that $w = c_1(t-1)e^{-t}$ and so $v = c_1 te^{-t} + c_2$, and therefore the general solution is

$$y(t) = c_1 t + c_2 e^t.$$

7. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are the fundamental solutions of

$$t^2 y'' - 2y = 0, \ t > 0.$$

   Solution: We can verify this by checking that the Wronskian is nonzero.

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0.$$

Chapter 4:

8. Find the solution to each:

(a) $y^{(3)} + 4y' = 0$

Solution: The characteristic equation is $r^3+4r = 0$. The roots are $r = 0, 2i, -2i$. Therefore the solution is

$$c_1 + c_2\cos(2t) + c_3\sin(2t).$$

(b) $y^{(4)} - 5y'' + 4y = e^t$

Solution: The characteristic equation is $r^4 - 5r^2 + 4 = 0$ which has roots $r = \pm 1, \pm 2$. Therefore the homogenous solution is

$$y_h(t) = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_3e^{-2t}.$$

Since e^t appears in our homogeneous solution, we should multiply by t to get a "guess" for the particular solution:

$$Y_P(t) = Ate^t.$$

The derivatives of this are: $Y_P'(t) = Ate^t + Ae_t$, $Y_P''(t) = Ate^t + 2Ae_t$, $Y_P'''(t) = Ate^t + 3Ae_t$, and $Y_P^{(4)}(t) = Ate^t + 4Ae_t$. Plugging in, we get

$$(Ate^t + 4Ae_t) - 5(Ate^t + 2Ae_t) + 4(Ate^t) = e^t.$$

Now, we find that 4A - 10A = 1, so $A = -1/6$. Therefore the solution is

$$y_h(t) = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_3e^{-2t} - \frac{1}{6}te^t.$$

9. Check whether the following are linearly independent or linearly dependent:

$$f_1(t) = 2t - 3, \;\; f_2(t) = 2t^2 + 1, \;\; f_3(t) = 3t^2 + t$$

Solution: If they are linearly dependent, there should be some k_1, k_2, k_3, so that

$$k_1(2t - 3) + k_2(2t^2 + 1) + k_3(3t^2 + t) = 0.$$

Plugging in $t = 0$, this means that

$$-3k_1 + k_2 = 0.$$

Plugging in $t = 1$,

$$-k_1 + 3k_2 + 4k_3 = 0.$$

Finally, plugging in $t = -1$,

$$-5k_1 + 3k_2 + 2k_3 = 0.$$

Solving this system of equations, we find that $k_1 = k_2 = k_3 = 0$ and so f_1, f_2, f_3 are linearly independent.

Chapter 7:

10. Find the solution to the system of equations:

$$x_1' = 3x_1 + 6x_2$$
$$x_2' = x_1 - 2x_2$$

with the initial conditions $x_1(0) = 0, x_2(0) = 1$.

Solution: First we find the eigenvalues of

$$\boldsymbol{A} = \begin{pmatrix} 3 & 6 \\ 1 & -2 \end{pmatrix}.$$

To do this, we set the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ equal to zero.

$$\begin{vmatrix} 3-\lambda & 6 \\ 1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6 = \lambda^2 - \lambda - 12 = 0.$$

Therefore $\lambda_1 = 4$ and $\lambda_2 = -3$. Next, we find the eigenvectors for each eigenvalue. For eigenvalue $\lambda_1 = 4$, we plug in and get:

$$\begin{pmatrix} -1 & 6 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $\eta_1 = 6\eta_2$. So an eigenvector for this is: $\boldsymbol{\eta} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Now, for $\lambda = -3$, we get the matrix formula:

$$\begin{pmatrix} 6 & 6 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $\eta_1 = -\eta_2$. So an eigenvector for this is: $\boldsymbol{\eta} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So, we can write the solution as:

$$\boldsymbol{x} = c_1 \begin{pmatrix} 6 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.$$

Finally, we solve for c_1 and c_2 using our initial conditions.

$$\boldsymbol{x}(0) = c_1 \begin{pmatrix} 6 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

And so, $6c_1 + c_2 = 0$ and $c_1 - c_2 = 1$. Therefore, $c_1 = 1/7$ and $c_2 = -6/7$. Plugging these in gives us our solution.

$$\boldsymbol{x} = \frac{1}{7} \begin{pmatrix} 6 \\ 1 \end{pmatrix} e^{4t} - \frac{6}{7} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.$$

11. Find the general solution to the system of equations:

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \boldsymbol{x}$$

Solution: First we find the eigenvalues of $\boldsymbol{A}$. To do this, we set the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ equal to zero.

$$\begin{vmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 10 = \lambda^2 + 9 = 0.$$

Therefore $\lambda = \pm 3i$. Next, we find an eigenvector for the eigenvalue $\lambda = 3i$.

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $(1-3i)\eta_1 = -2\eta_2$. So an eigenvector for this is: $\boldsymbol{\eta} = \begin{pmatrix} 1+3i \\ -5 \end{pmatrix}$. Therefore one solution (using Euler's formula) looks like:

$$\boldsymbol{x_1} = \begin{pmatrix} 1+3i \\ -5 \end{pmatrix} e^{3it} = \begin{pmatrix} 1+3i \\ -5 \end{pmatrix} (\cos(3t) + i\sin(3t)).$$

Separating real and complex parts, we get that this equals:

$$\boldsymbol{x_1} = \begin{pmatrix} \cos(3t) - 3\sin(3t) \\ -5\cos(3t) \end{pmatrix} + i \begin{pmatrix} 3\cos(3t) + \sin(3t) \\ -5\sin(3t) \end{pmatrix}.$$

The real and complex parts of this solution are in fact, fundamental solutions. Therefore, we can write the general solution as the linear combination of these:

$$\boldsymbol{x} = c_1 \begin{pmatrix} \cos(3t) - 3\sin(3t) \\ -5\cos(3t) \end{pmatrix} + c_2 \begin{pmatrix} 3\cos(3t) + \sin(3t) \\ -5\sin(3t) \end{pmatrix}.$$