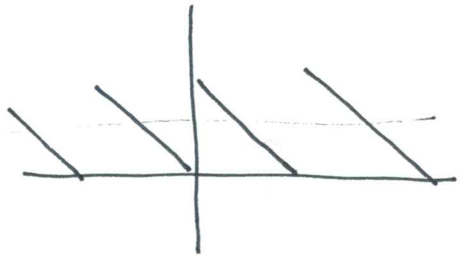
6. $f(x) = 1 - x$ on $0 \le x < 1$



$L = \frac{1}{2}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2n\pi x) + b_n \sin(2n\pi x)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 2\int_{-1/2}^{1/2} f(x)\,dx = 2\int_{-1/2}^{0}(-x)\,dx + 2\int_{0}^{1/2} 1 - x\,dx$$

$$= -x^2\Big|_{-1/2}^{0} + (2x - x^2)\Big|_{0}^{1/2} = \frac{1}{4} + 1 - \frac{1}{4} = 1$$

(<u>OR</u> take integral from 0 to 1 — easier)

$$a_n = 2\int_{-1/2}^{0}(-x)\cos(2n\pi x)\,dx + 2\int_{0}^{1/2}(1-x)\cos(2\pi n x)\,dx$$

$$= 2\left(\frac{1}{4\pi^2 n^2}\left[(-1)^n - 1\right)\right) + 2\left(\frac{1}{4\pi^2 n^2}\left(1 - (-1)^n\right)\right)$$

$$= 0$$

$$b_n = 2\int_{-1/2}^{0} (-x)\sin(2\pi n x)\,dx + 2\int_{0}^{1/2} (1-x)\sin(2\pi n x)\,dx$$

$$= 2\left(\frac{(-1)^n}{4\pi n} + \frac{1}{2\pi n} - \frac{1}{4\pi n}(-1)^n\right)$$

$$= \frac{1}{\pi n}$$

Solution is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin(2\pi n x)$$

8. $u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{4n^2 t}$ $\quad L = \pi$

where $u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = 1$ on $0 \leq x \leq \pi$.

so

$$c_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx$$

$$\frac{1}{\pi}\frac{\cos(nx)}{n}\Bigg|_{-\pi}^{0} + -\frac{\cos(nx)}{\pi n}\Bigg|_{0}^{\pi}$$

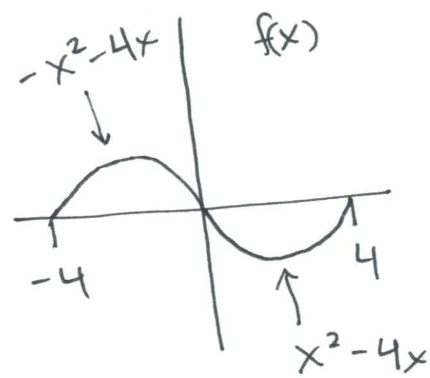
$$\frac{1}{\pi n} - \frac{(-1)^n}{\pi n} - \frac{(-1)^n}{\pi n} + \frac{1}{\pi n} = \frac{2}{\pi n}\left(1-(-1)^n\right)$$

so $u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n} \sin(nx) e^{4n^2 t}$

9. $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{4}\right)\left[c_n \cos\left(\frac{n\pi t}{4}\right) + d_n \sin\left(\frac{n\pi t}{4}\right)\right]$

$$\left.\begin{aligned} u(x,0) &= f(x) = x^2 - 4x \\ u_t(x,0) &= g(x) = 1 \end{aligned}\right\} \text{ on } 0 \le x \le 4$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{4}\right)$$



$$c_n = \frac{1}{4}\int_{-4}^{0} (-x^2-4x)\sin\left(\frac{n\pi x}{4}\right)dx + \frac{1}{4}\int_{0}^{4} (x^2-4x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{4}\left(\frac{64(2\cos(\pi n)-2)}{\pi^3 n^3}\right) + \frac{1}{4}\left(\frac{64(2\cos(\pi n)-2)}{\pi^3 n^3}\right)$$

$$= \frac{64}{\pi^3 n^3}\left((-1)^n - 1\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi}{4} d_n \sin\left(\frac{n\pi x}{4}\right)$$

$$\frac{n\pi}{4} d_n = \frac{1}{4}\int_{-4}^{0} -\sin\left(\frac{n\pi x}{4}\right)dx + \frac{1}{4}\int_{0}^{4} \sin\left(\frac{n\pi x}{4}\right)dx$$

$$= -\frac{1}{\cancel{4}}\left(\frac{\cancel{4}}{\pi n}\left(\cos(\pi n)-1\right)\right) + \frac{\cancel{4}}{\cancel{4}\pi n}\left(1-\cos(\pi n)\right)$$

$$= \frac{2}{\pi n}\left(1-(-1)^n\right)$$

$$d_n = \frac{4}{n\pi}\left(\frac{2}{\pi n}\right)\left(1-(-1)^n\right).$$